Phase-valued models of linear set theory

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Abstract

The aim of this paper is a model-theoretic study of the linear set theory. Following the standard practice in intuitionistic and quantum set theories, we define a set to be a function from its members to non-standard truth values. In our case, the truth values are facts in a phase space as defined by Girard. We will construct the universe $V^{\mathcal{P}}$ from the phase space $\mathcal{P}$ and verify a number of set-theoretic principles which are linear logic versions of the ZF axioms.

1 Introduction

In this paper, we will extend the Boolean-valued model for classical set theory [4, 7] to linear logic. This is in analogy to the locale (Heyting))-valued model for intuitionistic set theory [1], and, Takeuti and Titani's ortholattice-valued model for quantum set theory [6]. The general idea is as follows. Given a propositional logic and its algebraic model, we can regard an element of the algebra as a (non-standard) truth value. Then we can extend the notion of characteristic functions, or sets, so that their range becomes the set of the extended truth values.

In the case of linear logic, such an underlying set of truth values is given by the set of *facts* in a phase space as defined by Girard [3]. It is worth noting the similarity of the set of facts with the ortholattice in quantum logic. In short, the ortholattice is the lattice of closed subspaces of a Hilbert space ordered by inclusion. To each Hilbert space corresponds a physical system. Each vector in the space represents a state that a physical system can assume and each closed subspace represents a observable property of the physical system. Duals are defined by the orthogonality in the Hilbert space. Then, the correspondence is:

- phase space/Hilbert space
- facts/closed subspaces

- phase/vector

In fact, this is not at all surprising since Girard clearly modeled his phase spaces on Hilbert spaces. The point stating the similarity explicitly is to give the reader some assurance that the approach taken in quantum set theory can be transferred to linear set theory, at least to some extent.

2 Preliminary

In this section, we review phase space semantics for linear logic and the construction of Boolean-valued models.

Definition 2.1. *A phase space is a quadruple $\mathcal{P} = (P, 1, \cdot, \perp)$ where*

- *$(P, 1, \cdot)$ is a commutative monoid*
- *$\perp \subset P$.*

We will abbreviate $p \cdot q$ by pq.

Definition 2.2. *Given a subset A of P, the* dual *of A, denoted $A^\perp$, is defined by $A^\perp = \{p \in P \mid (\forall q \in A)\ pq \in \perp\}$.*

Definition 2.3. *A subset A of P is a fact if $A = A^{\perp\perp}$. We denote the set of facts in $\mathcal{P}$ by $FACT_{\mathcal{P}}$.*

Definition 2.4. *A fact A is valid if $1 \in A$.*

Proposition 2.5. *Facts are closed under arbitrary intersection.*

Proof. First of all, $A \subset A^{\perp\perp}$. So. we only need to show the other direction. Let $\{F_i\}_{i \in I}$ be a family of facts and $A = \bigcap F_i$. Suppose $p \in A^{\perp\perp}$. We want to show $p \in F_i^{\perp\perp}$ for all $i \in I$. Let $q \in F_i^\perp$. Take any $r \in A$. Then $r \in F_i$. So, $qr \in \perp$. Therefore, $q \in A^\perp$. Hence, $pq \in \perp$. That is to say, $p \in F_i^{\perp\perp}$. □

Proposition 2.6. *Given a subset A of P, the set $A^{\perp\perp}$ is the smallest fact containing A.*

Proof. First, we show $A^\perp = A^{\perp\perp\perp}$. One direction is trivial. So let $p \in A^{\perp\perp\perp}$. Take any $q \in A$. Then, $q \in A^{\perp\perp}$. So, $pq \in \perp$. Then $A^{\perp\perp\perp\perp} = A^{\perp\perp}$. Hence $A^{\perp\perp}$ is a fact.

Now suppose $A \subset B$ and B is a fact. Let $p \in B^\perp$ and take any $q \in A$. Then $q \in B$ so that $pq \in \perp$. Hence $B^\perp \subset A^\perp$. Then, by the same argument, $A^{\perp\perp} \subset B^{\perp\perp} = B$. □

Definition 2.7. *We define multiplicative operations on the set of facts in $\mathcal{P}$ as follows:*

- $F \otimes G = (FG)^{\perp\perp}$
- $F \invamp G = (F^\perp G^\perp)^\perp$

- $F \multimap G = (FG^{\perp})^{\perp}$

where F and G are facts in $\mathcal{P}$ and $FG = \{pq \mid p \in F \text{ and } q \in G\}$.

Definition 2.8. *We define additive operations on the set of facts in $\mathcal{P}$ as follows:*

- $F \& G = F \cap G$
- $F \oplus G = (F \cup G)^{\perp\perp}$

Definition 2.9. *We define constants in the set of facts in $\mathcal{P}$ as follows:*

- $\mathbf{1} = \perp^{\perp}$
- $\top = \emptyset^{\perp}$
- $\mathbf{0} = \top^{\perp}$

Note that $\perp$ and all other constants are facts in $\mathcal{P}$.

Now we define semantics for the multiplicative-additive fragment of linear logic (**MALL**).

Definition 2.10. *A phase structure for **MALL** is a phase space with a function which assigns a fact to each propositional letter. The interpretation of a sentence is a fact assigned to the sentence by extending the function inductively.*

Definition 2.11. *A sentence is valid if the identity 1 is in its interpretation. A sentence is a linear tautology if it is valid in any phase structure.*

Proposition 2.12. **MALL** *is sound and complete with respect to the validity in phase structure.*

For the proof of the proposition, we refer the reader to Girard's original paper [3].

The phase semantics can be easily extended to predicate logic. We simply interpret quantifications as infinitary *additive* conjunction $\bigcap F_i$ and and disjunction $(\bigcup F_i)^{\perp\perp}$. For exponentials, we need to extend the phase space.

Definition 2.13. *A topolinear space is a phase space paired with the set $\mathcal{F}$ of the* closed facts *such that:*

(i) *$\mathcal{F}$ is closed under arbitrary intersection (additive conjunction)*

(ii) *$\mathcal{F}$ is closed under finite multiplicative disjunction*

(iii) *$\perp$ is the smallest fact in $\mathcal{F}$*

(iv) *For all $A \in \mathcal{F}$, $A \parr A = A$.*

The linear negations of closed facts are called open facts.

Definition 2.14. *We define the exponential operations on the set of facts as follows:*

- $!F =$ *the greatest open fact included in* F
- $?F =$ *the smallest closed fact containing* F

where the order is with respect to set inclusion.

There is a new simplified version of the definition of exponentials in the phase space. For our present purpose, however, the above definition suffices.

Proposition 2.15. *Linear logic is sound and complete with respect to the validity in the topolinear spaces.*

The following propositions are useful.

Proposition 2.16. *Let* F *and* G *be facts in* $\mathcal{P}$. *Then,* $1 \in F \multimap G$ *if and only if* $F \subset G$.

Proof. First, assume $1 \in F \multimap G = (F \cdot G^{\perp})^{\perp}$. Then $\{1\} \cdot (FG^{\perp}) = FG^{\perp} \in \perp$. Hence $F \subset G^{\perp\perp} = G$. Secondly, assume $F \subset G$. Then $G^{\perp} \subset F^{\perp}$ so that $FG^{\perp} \subset FF^{\perp} \subset \perp$. Hence, $\{1\} \cdot (FG^{\perp}) \subset \perp$. Therefore $1 \in (FG^{\perp})^{\perp} = F \multimap G$. $\square$

Proposition 2.17. *Let* F_i *and* G *be facts. Then* $(\bigcup F_i)^{\perp\perp} \otimes G = (\bigcup F_i \otimes G)^{\perp\perp}$.

Proof. First, let $p \in F_i G$. Then $p = qr$ for some $q \in F_i \subset (\bigcup F_i)^{\perp\perp}$ and $r \in G$. Hence $qr \in (\bigcup F_i)^{\perp\perp} \otimes G$. By the proposition 2.6, we can conclude that $(\bigcup F_i \otimes G)^{\perp\perp} \subset (\bigcup F_i)^{\perp\perp} \otimes G$.

Secondly, we show $(\bigcup F_i)^{\perp\perp} \cdot G \subset (\bigcup F_i \otimes G)^{\perp\perp}$. Let $p \in G$ and $q \in (\bigcup F_i \otimes G)^{\perp}$. Let $r \in F_i$ for some i. Then $rp \in \bigcup F_i \otimes G$ so that $pqr \in \perp$. Therefore $pq \in (\bigcup F_i)^{\perp}$. Hence $(\bigcup F_i)^{\perp\perp} \cdot G \cdot \{q\} \subset \perp$, *i.e.*, $(\bigcup F_i)^{\perp\perp} \cdot G \subset (\bigcup F_i \otimes G)^{\perp\perp}$. Therefore $(\bigcup F_i)^{\perp\perp} \otimes G \subset (\bigcup F_i \otimes G)^{\perp\perp}$. $\square$

Proposition 2.18. *Let* F, G *and* H *be facts. Then* $(F \otimes G) \otimes H = (F \cdot G \cdot H)^{\perp\perp}$.

Proof. First, $FGH \subset (F \otimes G) \cdot H \subset (F \otimes G) \otimes H$. Hence $(FGH)^{\perp\perp} \subset (F \otimes G) \otimes H$.

On the other hand, let $p \in H$ and $q \in (FGH)^{\perp}$. Take any $r \in FG$. Then $pqr \in \perp$ and $pq \in (FG)^{\perp}$. Hence, $(F \otimes G) \cdot G \cdot \{q\} \subset \perp$. Therefore $q \in ((F \otimes G) \cdot G)^{\perp}$. From this, it follows that $(F \otimes G) \otimes H \subset (FGH)^{\perp\perp}$. $\square$

Proposition 2.19. *Let* F_i *be facts. Then* $(\bigcap F_i^{\perp})^{\perp} = (\bigcup F_i)^{\perp\perp}$.

Proof. It suffices to show that $\bigcap F_i^{\perp} = (\bigcup F_i)^{\perp}$. Let $p \in \bigcap F_i^{\perp}$ and $q \in F_i$ for some i. Then $pq \in \perp$. Hence $p \in (\bigcup F_i)^{\perp}$. For the other direction, let $p \in (\bigcup F_i)^{\perp}$ and $q \in F_i \subset \bigcup F_i$. Then $pq \in \perp$. So $p \in F_i^{\perp}$. Hence $p \in \bigcap F_i^{\perp}$. $\square$

We now turn our attention to Boolean-valued models. Let $\mathcal{B}$ be a complete Boolean algebra. We first define the $\mathcal{B}$-valued universe $V^{\mathcal{B}}$.

Definition 2.20. *We define* $V_\alpha^{\mathcal{B}}$ *and* $V^{\mathcal{B}}$ *by the transfinite induction on ordinals* α *as follows:*

- $V_0^{\mathcal{B}} = \emptyset$
- $V_{\alpha+1}^{\mathcal{B}} = \{u \mid u \text{ is a function with } dom(u) \subset V_a^{\mathcal{B}} \text{ and } ran(u) = \mathcal{B}\}$
- $V_\lambda^{\mathcal{B}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathcal{B}}$ *where* λ *is a limit ordinal.*
- $V^{\mathcal{B}} = \bigcup_{\alpha \in ON} V_\alpha^{\mathcal{B}}$.

Next we define the interpretation of "atomic propositions". Note that we can assign the rank $\rho(u)$ to each $u \in V^{\mathcal{B}}$ by defining:

$$\rho(u) = \text{ the least } \alpha \text{ such that } u \in V_{\alpha+1}^{\mathcal{B}}.$$

Definition 2.21. *For* $u, v \in V^{\mathcal{B}}$*, we define* $[\![u = v]\!], [\![u \subset v]\!]$ *and* $[\![u \in v]\!]$ *by transfinite induction on* $(\rho(u), \rho(v))$ *as follows:*

- $[\![u \in v]\!] = \bigvee_{x \in dom(v)} (v(x) \wedge [\![x = u]\!])$
- $[\![u \subset v]\!] = \bigwedge_{x \in dom(u)} (u(x) \to [\![x \in v]\!])$ *where* $a \to b = \neg a \vee b$
- $[\![u = v]\!] = [\![u \subset v]\!] \wedge [\![v \subset u]\!]$

The idea behind the above definition is the following translation:

- $u \in v \iff (\exists x \in v)(x = u)$
- $u \subset v \iff (\forall x \in u)(x \in v)$
- $u = v \iff u \subset v$ *and* $v \subset u$

Notice that universal and existential quantifications are interpreted as infinitary conjunction (meet) and disjunction (join) respectively.

Proposition 2.22. *For every* $u, v \in V^{\mathcal{B}}$,

(i) $[\![u = u]\!] = 1$

(ii) $[\![u = v]\!] = [\![v = u]\!]$

(iii) $[\![u = v]\!] \wedge [\![v = w]\!] \le [\![u = w]\!]$

(iv) $[\![u \in v]\!] \wedge [\![w = u]\!] \wedge [\![t = v]\!] \le [\![w \in t]\!]$

The proof is by the induction on ranks. Now we extend this assignment to every sentence.

Definition 2.23. *For every formula* $\varphi(x_1, \ldots, x_n)$*, we define the* Boolean value *of* φ

$$[\![\varphi(u_1, \ldots, u_n)]\!] \qquad\qquad (u_1, \ldots, u_n \in V^{\mathcal{B}})$$

as follows:

(a) *If φ is an atomic formula, the assignment is as we defined above*

(b) *If φ is a negation, conjunction, etc.,*

$$\begin{aligned}
[\![\neg\psi(u_1,\dots,u_n)]\!] &= \neg[\![\psi(u_1,\dots,u_n)]\!] \\
[\![\psi\wedge\chi(u_1,\dots,u_n)]\!] &= [\![\psi(u_1,\dots,u_n)]\!]\wedge[\![\chi(u_1,\dots,u_n)]\!] \\
[\![\psi\vee\chi(u_1,\dots,u_n)]\!] &= [\![\psi(u_1,\dots,u_n)]\!]\vee[\![\chi(u_1,\dots,u_n)]\!] \\
[\![\psi\to\chi(u_1,\dots,u_n)]\!] &= [\![\psi(u_1,\dots,u_n)]\!]\to[\![\chi(u_1,\dots,u_n)]\!] \\
[\![\psi\leftrightarrow\chi(u_1,\dots,u_n)]\!] &= [\![\psi\to\chi(u_1,\dots,u_n)]\!]\wedge[\![\chi\to\psi(u_1,\dots,u_n)]\!]
\end{aligned}$$

(c) *If φ is $\exists x\psi$ or $\forall x\psi$*

$$\begin{aligned}
[\![\exists x\psi(x,u_1,\dots,u_n]\!] &= \bigvee_{v\in V^{\mathcal{B}}}[\![\psi(v,u_1,\dots,u_n]\!] \\
[\![\forall x\psi(x,u_1,\dots,u_n]\!] &= \bigwedge_{v\in V^{\mathcal{B}}}[\![\psi(v,u_1,\dots,u_n]\!]
\end{aligned}$$

Definition 2.24. *A sentence φ is* valid *in $V^{\mathcal{B}}$ if $[\![\varphi]\!]=1$.*

Proposition 2.25. *Every axiom of ZFC is valid in $V^{\mathcal{B}}$.*

3 The Phase-valued Model $V^{\mathcal{P}}$

We now define our first model $V^{\mathcal{P}}$. The construction is essentially the same as that of $V^{\mathcal{B}}$ except that we will use the set of facts in a phase space instead of the boolean algebra.

Definition 3.1. *We define $V^{\mathcal{P}}_\alpha$ and $V^{\mathcal{P}}$ by the transfinite induction on ordinals α as follows:*

- $V^{\mathcal{P}}_0=\emptyset$
- $V^{\mathcal{P}}_{\alpha+1}=\{u\mid u$ *is a function with* $dom(u)\subset V^{\mathcal{P}}_a$ *and* $ran(u)=FACT_{\mathcal{P}}\}$
- $V^{\mathcal{P}}_\lambda=\bigcup_{\alpha<\lambda}V^{\mathcal{P}}_\alpha$ *where λ is a limit ordinal.*
- $V^{\mathcal{P}}=\bigcup_{\alpha\in ON}V^{\mathcal{P}}_\alpha$.

Proposition 3.2. *$V^{\mathcal{P}}_\beta\subset V^{\mathcal{P}}_\alpha$ for $\beta<\alpha$.*

Proof. The proof is by transfinite induction on α. Assume that $V^{\mathcal{P}}_\gamma\subset V^{\mathcal{P}}_\beta$ holds for any $\gamma<\beta$ with $\beta<\alpha$. If α is a limit ordinal, then the proposition holds by the definition. Suppose that $\alpha=\alpha'+1$. Let $\rho(f)=\beta$ with $\beta<\alpha$. Then β is a successor $\beta'+1$ and $dom(f)\subset V^{\mathcal{P}}_{\beta'}$. Then $\beta'<\alpha'$ and $V^{\mathcal{P}}_{\beta'}\subset V^{\mathcal{P}}_{\alpha'}$ by the inductive hypothesis. Hence $f\in V^{\mathcal{P}}_\alpha$. □

Definition 3.3. *For $u,v\in V^{\mathcal{P}}$, we define $[\![u=v]\!],[\![u\subset v]\!]$ and $[\![u\in v]\!]$ by transfinite induction on $(\rho(u),\rho(v))$ as follows:*

- $[\![u \in v]\!] = (\bigcup_{x \in dom(v)} v(x) \otimes [\![x = u]\!])^{\perp\perp}$
- $[\![u \subset v]\!] = \bigcap_{x \in dom(u)} (u(x) \multimap [\![x \in v]\!])$
- $[\![u = v]\!] = ![\![u \subset v]\!] \otimes ![\![v \subset u]\!]$

Proposition 3.4. *For every $u, v \in V^{\mathcal{P}}$,*

(i) $1 \in u(x) \multimap [\![x \in u]\!]$ *for all $x \in dom(u)$*

(ii) $1 \in [\![u = u]\!]$

(iii) $1 \in [\![u = v]\!] \multimap [\![v = u]\!]$

Proof. We prove (i) and (ii) together by the simultaneous induction on ranks.

(i) It suffices to show that for all $x \in dom(u)$,

$$u(x) \cdot [\![x \in u]\!]^{\perp} \subset \perp.$$

By the inductive hypothesis, $1 \in [\![x = x]\!]$. Then $u(x) = u(x) \cdot \{1\} \subset (u(x) \cdot \{1\})^{\perp\perp} \subset \bigcup_{y \in dom(u)} (u(y) \cdot [\![x = y]\!])^{\perp\perp}$. Hence $u(x) \cdot [\![x \in u]\!]^{\perp} = u(x) \cdot (\bigcup_{y \in dom(u)} u(y) \otimes [\![x = y]\!])^{\perp\perp\perp} = u(x) \cdot (\bigcup_{y \in dom(u)} u(y) \otimes [\![x = y]\!])^{\perp} \subset \perp$.

(ii) It suffices to show $1 \in ![\![u \subset u]\!]$. Now $[\![u \subset u]\!] = \bigcap_{x \in dom(u)} (u(x) \multimap [\![x \in u]\!])$. By (i), we have $u(x) \cdot [\![x \in u]\!]^{\perp} \subset \perp$ for all $x \in dom(u)$ so that $\mathbf{1} = \perp^{\perp} \subset \bigcap_{x \in dom(u)} (u(x) \multimap [\![x \in u]\!])$. Since $\perp$ is the smallest closed fact, $\mathbf{1}$ is the greatest open fact. Hence $\mathbf{1} = ![\![u \subset u]\!]$ and clearly $1 \in \mathbf{1}$.

(iii) We need to show that $1 \in ([\![u = v]\!] \cdot [\![v = u]\!]^{\perp})^{\perp}$. Now

$$1 \in ([\![u = v]\!] \cdot [\![v = u]\!]^{\perp})^{\perp} \iff [\![u = v]\!] \cdot [\![v = u]\!]^{\perp} \subset \perp .$$

Since $[\![u = v]\!] = ![\![u \subset v]\!] \otimes ![\![v \subset u]\!] = ![\![v \subset u]\!] \otimes ![\![u \subset v]\!] = [\![v = u]\!]$, we have $[\![u = v]\!] \cdot [\![v = u]\!]^{\perp} = [\![u = v]\!] \cdot [\![u = v]\!]^{\perp} \subset \perp$.

□

Proposition 3.5. *For every $u, v, w \in V^{\mathcal{P}}$,*

(i) $1 \in [\![u = v]\!] \otimes [\![v = w]\!] \multimap [\![u = w]\!]$

(ii) $1 \in [\![u \in v]\!] \otimes [\![u = w]\!] \multimap [\![w \in v]\!]$

(iii) $1 \in [\![u \in v]\!] \otimes [\![v = w]\!] \multimap [\![u \in w]\!]$

Proof. The proof is by the simultaneous induction on the canonical ordering [4] of $(\rho(u), \rho(v), \rho(w))$.

(i) First we show $[\![u \subset v]\!] \otimes [\![v = w]\!] \subset [\![u \subset w]\!]$. Now

$$\begin{aligned}
[\![u \subset v]\!] \otimes [\![v = w]\!] &= (\bigcap_{x \in dom(u)} (u(x) \cdot [\![x \in v]\!]^{\perp})^{\perp}) \otimes [\![v = w]\!] \\
&= ((\bigcap_{x \in dom(u)} (u(x) \cdot [\![x \in v]\!]^{\perp})^{\perp}) \cdot [\![v = w]\!])^{\perp\perp}
\end{aligned}$$

Let $p \in \bigcap_{x \in dom(u)} (u(x) \cdot [\![x \in v]\!]^{\perp})^{\perp}$. Then for all $x \in dom(u), s \in u(x)$ and $s' \in [\![x \in v]\!]^{\perp}$, we have $pss' \in \perp$ so that $ps \in [\![x \in v]\!]^{\perp\perp} = [\![x \in v]\!]$.

Now we show that for any $q \in [\![v = w]\!]$, we have $pq \in \bigcap_{x \in dom(u)} (u(x) \cdot [\![x \in w]\!]^{\perp})^{\perp}$. Fix $y \in dom(u)$ and let $r \in u(y) \cdot [\![y \in w]\!]^{\perp}$. Then $r = tt'$ where $t \in u(y)$ and $t' \in [\![y \in w]\!]^{\perp}$. Then $pt \in [\![y \in v]\!]$. By the inductive hypothesis, $[\![y \in v]\!] \cdot [\![v = w]\!] \subset [\![y \in v]\!] \otimes [\![v = w]\!] \subset [\![y \in w]\!]$. Hence $ptq \in [\![y \in w]\!]$. Therefore $pqr = ptqt' \in \perp$.

Similarly, we can show $[\![w \subset v]\!] \otimes [\![u = v]\!] \subset [\![w \subset u]\!]$.

Next we show $[\![u = v]\!] \otimes [\![v = w]\!] \subset [\![u = w]\!]$. Note that $!A \subset A$ since $!A$ is the greatest open fact included in A. Also for any open fact A, we have $A \otimes A = A$ since closed facts are idempotent with respect to multiplicative disjunction. Then

$$\begin{aligned}
[\![u = v]\!] \otimes [\![v = w]\!] &= ![\![u \subset v]\!] \otimes ![\![v \subset u]\!] \otimes ![\![v \subset w]\!] \otimes ![\![w \subset v]\!] \\
&= ![\![u \subset v]\!] \otimes ![\![u \subset v]\!] \otimes ![\![v \subset u]\!] \\
&\qquad \otimes ![\![w \subset v]\!] \otimes ![\![v \subset w]\!] \otimes ![\![w \subset v]\!] \\
&= ![\![u \subset v]\!] \otimes [\![w = v]\!] \otimes ![\![w \subset v]\!] \otimes [\![u = v]\!]
\end{aligned}$$

Now $![\![u \subset v]\!] \otimes [\![w = v]\!] \subset [\![u \subset v]\!] \otimes [\![w = v]\!] \subset [\![u \subset w]\!]$. Since open sets are closed under finite multiplicative conjunction, $![\![u \subset v]\!] \otimes [\![w = v]\!]$ is open. Therefore, we have $![\![u \subset v]\!] \otimes [\![w = v]\!] \subset ![\![u \subset w]\!]$. Similarly, $![\![w \subset v]\!] \otimes [\![u = v]\!] \subset ![\![w \subset u]\!]$. Hence,$[\![u = v]\!] \otimes [\![v = w]\!] \subset [\![u = w]\!]$.

(ii) We want to show $[\![u \in v]\!] \otimes [\![u = w]\!] \subset [\![w \in v]\!]$. Now

$$\begin{aligned}
[\![u \in v]\!] \otimes [\![u = w]\!] &= ((\bigcup_{x \in dom(v)} (v(x) \cdot [\![x = u]\!])^{\perp\perp})^{\perp\perp} \cdot [\![u = w]\!])^{\perp\perp} \\
&= (\bigcup_{x \in dom(v)} (v(x) \cdot [\![x = u]\!] \cdot [\![u = w]\!])^{\perp\perp})^{\perp\perp}
\end{aligned}$$

By the inductive hypothesis,

$$[\![x = u]\!] \cdot [\![u = w]\!] \subset [\![x = u]\!] \otimes [\![u = w]\!] \subset [\![x = w]\!].$$

Hence $[\![u \in v]\!] \otimes [\![v = w]\!] \subset (\bigcup_{x \in dom(v)} (v(x) \cdot [\![x = w]\!])^{\perp\perp})^{\perp\perp} = [\![w \in v]\!]$.

(iii) We want to show $[\![u \in v]\!] \otimes [\![v = w]\!] \subset [\![u \in w]\!]$. We know

$$\begin{aligned}[\![u \in v]\!] \otimes [\![v = w]\!] &= ((\bigcup_{x \in dom(v)} (v(x) \cdot [\![x = u]\!])^{\perp\perp})^{\perp\perp} \cdot [\![v = w]\!])^{\perp\perp} \\ &= (\bigcup_{x \in dom(v)} (v(x) \cdot ![\![v \subset w]\!] \cdot ![\![w \subset v]\!] \cdot [\![x = u]\!])^{\perp\perp})^{\perp\perp}\end{aligned}$$

Now fix $y \in dom(v)$. We have

$$\begin{aligned}v(y) \cdot ![\![v \subset w]\!] &\subset v(y) \cdot [\![v \subset w]\!] \\ &= v(y) \cdot (\bigcap_{x \in dom(v)} v(x) \multimap [\![x \in w]\!])\end{aligned}$$

Note that this is a subset of $[\![y \in w]\!]$. Let $q \in \bigcap_{x \in dom(v)}(v(x) \cdot [\![x \in w]\!]^{\perp})^{\perp}$. Then for any $r \in v(y)$ and $s \in [\![y \in w]\!]^{\perp}$, we have $qrs \in \perp$ so that $qr \in [\![y \in w]\!]^{\perp\perp} = [\![y \in w]\!]$. Hence $v(y) \cdot [\![v \subset w]\!] \subset [\![y \in w]\!]$.

Therefore it suffices to show

$$[\![y \in w]\!] \cdot ![\![w \subset v]\!] \cdot [\![y = u]\!] \subset (\bigcup_{z \in dom(w)} (w(z) \cdot [\![z = u]\!])^{\perp\perp})^{\perp\perp}.$$

By the inductive hypothesis, $[\![z = y]\!] \cdot [\![y = u]\!] \subset [\![z = y]\!] \otimes [\![y = u]\!] \subset [\![z = u]\!]$ for all $z \in dom(w)$. Then

$$\begin{aligned}[\![y \in w]\!] \cdot [\![y = u]\!] &= (\bigcup_{z \in dom(w)} (w(z) \cdot [\![z = y]\!])^{\perp\perp})^{\perp\perp} \cdot [\![y = u]\!] \\ &\subset (\bigcup_{z \in dom(w)} (w(z) \cdot [\![z = y]\!] \cdot [\![y = u]\!])^{\perp\perp})^{\perp\perp} \\ &\subset (\bigcup_{z \in dom(w)} (w(z) \cdot [\![z = u]\!])^{\perp\perp})^{\perp\perp} \\ &= [\![u \in w]\!]\end{aligned}$$

Now we show that for any open set C, if $A \subset B$, then $A \cdot C \subset B$. Assume $A \subset B$. Then $A \cdot B^{\perp} \subset \perp$. Then $A \cdot C \cdot B^{\perp} \subset \perp \cdot C$. So we want to show $\perp \cdot C \subset \perp$. Let $p \in \perp$ and $q \in C = D^{\perp}$ where D is closed. Then for any $r \in D$, we have $qr \in \perp$. In particular, $p \in \perp \subset D$ so that $pq \in \perp$. Hence $[\![y \in w]\!] \cdot ![\![w \subset v]\!] \cdot [\![y = u]\!] \subset (\bigcup_{z \in dom(w)}(w(z) \cdot [\![z = u]\!])^{\perp\perp})^{\perp\perp}$.

□

Definition 3.6. *For every formula $\varphi(x_1, \dots, x_n)$, we define the* phase value *of φ*

$$[\![\varphi(u_1, \dots, u_n)]\!] \qquad (u_1, \dots, u_n \in V^{\mathcal{P}})$$

as follows:

(a) *If φ is an atomic formula, the assignment is as we defined above*

(b) *If φ is a linear negation, multiplicative conjunction, etc.,*

$$\begin{array}{rcl}
[\![\psi(u_1,\dots,u_n^{\perp})]\!] & = & [\![\psi(u_1,\dots,u_n)]\!]^{\perp} \\
[\![\psi\otimes\chi(u_1,\dots,u_n]\!] & = & ([\![\psi(u_1,\dots,u_n)]\!]\cdot[\![\chi(u_1,\dots,u_n)]\!])^{\perp\perp} \\
[\![\psi \parr \chi(u_1,\dots,u_n]\!] & = & ([\![\psi(u_1,\dots,u_n)]\!]^{\perp}\cdot[\![\chi(u_1,\dots,u_n)]\!]^{\perp})^{\perp} \\
[\![\psi\multimap\chi(u_1,\dots,u_n)]\!] & = & ([\![\psi(u_1,\dots,u_n)]\!]\cdot[\![\chi(u_1,\dots,u_n)]\!]^{\perp})^{\perp} \\
[\![\psi\&\chi(u_1,\dots,u_n]\!] & = & [\![\psi(u_1,\dots,u_n)]\!]\cap[\![\chi(u_1,\dots,u_n)]\!] \\
[\![\psi\oplus\chi(u_1,\dots,u_n]\!] & = & ([\![\psi(u_1,\dots,u_n)]\!]\cup[\![\chi(u_1,\dots,u_n)]\!])^{\perp\perp}
\end{array}$$

(c) *If φ is $\exists x\psi$ or $\forall x\psi$,*

$$\begin{array}{rcl}
[\![\exists x\psi(x,u_1,\dots,u_n)]\!] & = & (\bigcup_{v\in V^{\mathcal{B}}}[\![\psi(v,u_1,\dots,u_n)]\!])^{\perp\perp} \\
[\![\forall x\psi(x,u_1,\dots,u_n)]\!] & = & \bigcap_{v\in V^{\mathcal{B}}}[\![\psi(v,u_1,\dots,u_n)]\!]
\end{array}$$

(d) *If φ is $!\psi$ or $?\psi$,*

$$\begin{array}{rcl}
[\![!\psi(u_1,\dots,u_n)]\!] & = & ![\![\psi(u_1,\dots,u_n)]\!] \\
[\![?\psi(u_1,\dots,u_n)]\!] & = & ?[\![\psi(u_1,\dots,u_n)]\!]
\end{array}$$

Proposition 3.7. $1\in[\![u=v]\!]\otimes[\![\phi(u)]\!]\multimap[\![\phi(v)]\!]$ *for any formula ϕ.*

Proof. The proof is by induction on the construction of ϕ, using that $[\![u=v]\!]$ is an open fact. ☐

We now start checking the validity of the basic set-theoretical principles.

Proposition 3.8. (a) $[\![(\exists y\in x)\phi(y)]\!]=(\bigcup_{y\in dom(x)}(x(y)\otimes[\![\phi(y)]\!]))^{\perp\perp}$

(b) $[\![(\forall y\in x)\phi(y)]\!]=\bigcap_{y\in dom(x)}(x(y)\multimap[\![\phi(y)]\!])$

Proof. **(a)**

$$\begin{array}{rcl}
[\![(\exists y\in x)\phi(y)]\!] & = & [\![\exists y(y\in x\otimes\phi(y)]\!] \\
& = & (\bigcup_{y\in V^P}[\![y\in x\otimes\phi(y)]\!])^{\perp\perp} \\
& = & (\bigcup_{y\in V^P}([\![y\in x]\!]\otimes[\![\phi(y)]\!]))^{\perp\perp} \\
& = & (\bigcup_{y\in V^P}[(\bigcup_{z\in dom(x)}(x(z)\otimes[\![y=z]\!]))^{\perp\perp}\otimes[\![\phi(y)]\!]])^{\perp\perp} \\
& \subset & (\bigcup_{y\in V^P}[(\bigcup_{z\in dom(x)}(x(z)\otimes[\![y=z]\!]\otimes[\![\phi(y)]\!]))^{\perp\perp}])^{\perp\perp} \\
& \subset & (\bigcup_{z\in dom(x)}(x(z)\otimes[\![\phi(z)]\!]))^{\perp\perp}
\end{array}$$

Also,

$$
\begin{aligned}
\Big(\bigcup_{y\in dom(x)} (x(y)\otimes [\![\phi(y)]\!])\Big)^{\perp\perp} &\subset \Big(\bigcup_{y\in dom(x)} ([\![y\in x]\!]\otimes [\![\phi(y)]\!])\Big)^{\perp\perp}\\
&\subset \Big(\bigcup_{y\in V^P} ([\![y\in x]\!]\otimes [\![\phi(y)]\!])\Big)^{\perp\perp}\\
&= [\![\exists y(y\in x\otimes \phi(y)]\!]
\end{aligned}
$$

(b) The proof is the proposition 2.19 and **(a)**.

□

Now, we verify a number of formulas which are the linear logic counterparts of the ZF axioms.

Theorem 3.9. *The following formulas are valid in* $V^{\mathcal{P}}$*.*

(Empty Set): $\exists Y\forall x(x\in Y)^{\perp}$

(Extensionality):
$\forall X\forall Y(!\forall u(u\in X \multimap u\in Y)\otimes !\forall u(u\in Y\multimap u\in X)\multimap X=Y)$

(Pair): $\forall u\forall v\exists a\forall x(x=u\oplus x=v\multimap x\in a)$

(Union): $\forall X\exists Y\forall u(\exists z(z\in X\otimes u\in z)\multimap u\in Y)$

(Separation):
$\forall X\exists Y(!\,\forall u(u\in Y\multimap u\in X\otimes\phi(u))\otimes !\,\forall u(u\in X\otimes\phi(u)\multimap u\in Y))$

(Collection): $\forall u(\forall x\in u\exists y\phi(x,y)\multimap \exists v\forall x\in u\exists y\in v\phi(x,y))$

(Infinity): $\exists Y(!\,\emptyset_{\mathcal{P}}\in Y\otimes !\,\forall x(x\in Y\multimap x\cup\{x\}\in Y))$

Proof.

(Empty Set): Let $Y\in V^{\mathcal{P}}$ be such that $dom(Y)=\emptyset$. Then, for any $x\in V^{\mathcal{P}}$, we have $\bigcup_{v\in dom(Y)}(Y(v)\otimes[\![x=v]\!])=\emptyset$. Then

$$\Big(\bigcup_{v\in dom(Y)} (Y(v)\otimes [\![x=v]\!])\Big)^{\perp} = \top = P$$

Hence, $[\![(x\in Y)^{\perp}]\!]=P$ and $[\![\forall x(x\in Y)^{\perp}]\!]=P$. Obviously, $1\in P$.

(Extensionality): By the definition, $[\![X=Y]\!]=![\![X\subset Y]\!]\otimes ![\![Y\subset X]\!]$. Then, the axiom holds by the proposition 3.8.

(Pair): Let $a\in V^{\mathcal{P}}$ be such that $dom(a)=\{u,v\}$ and $1\in a(u)=a(v)$. Then, $1\in[\![u\in a]\!]$ and $1\in[\![v\in a]\!]$. Now for any $x\in V^{\mathcal{P}}$, we have $[\![x=u]\!]\otimes[\![u\in a]\!]\subset[\![x\in a]\!]$ so that $[\![x=u]\!]\subset[\![x\in a]\!]$. Similarly, $[\![x=v]\!]\subset[\![x\in a]\!]$. Hence $[\![x=u\oplus x=v]\!]=([\![x=u]\!]\cup[\![x=v]\!])^{\perp\perp}\subset[\![x\in a]\!]^{\perp\perp}=[\![x\in a]\!]$.

(Union): Let $Y \in V^{\mathcal{P}}$ be such that $dom(Y) = \bigcup\{dom(z) \mid z \in dom(X)\}$ and $Y(u) = (\bigcup_{z\in\{z|u\in dom(z) \ and \ z\in dom(X)\}} X(z) \otimes z(u))^{\perp\perp}$. Then for any $z \in dom(X)$ and $u \in dom(z)$, we have $X(z) \otimes z(u) \subset Y(u) \subset [\![u \in Y]\!]$ so that $X(z) \subset z(u) \multimap [\![u \in Y]\!]$. Hence, $X(z) \subset [\![\forall w(w \in z \multimap w \in Y)]\!]$. This means that $1 \in [\![\forall z(z \in X \multimap \forall w(w \in z \multimap w \in Y))]\!]$, which is equivalent to the validity of the axiom.

(Separation): Let $Y \in V^{\mathcal{P}}$ be such that $dom(Y) = dom(X)$ and $Y(u) = X(u)\otimes[\![\varphi(u)]\!]$ for all $u \in dom(Y)$. Then $Y(u) \subset [\![u \in X \otimes \varphi(u)]\!]$ for all $u \in dom(Y)$ so that $1 \in [\![\forall u(u \in Y \multimap u \in X \otimes \varphi(u))]\!]$. Also, $X(u) \otimes [\![\varphi(u)]\!] \subset Y(u) \subset [\![u \in Y]\!]$ for all $u \in dom(X)$ so that $X(u) \subset [\![\varphi(u)]\!] \multimap [\![u \in Y]\!]$. So $1 \in [\![\forall u(u \in X \multimap (\varphi(u) \multimap u \in Y))]\!] = [\![\forall u(u \in X \otimes \varphi(u) \multimap u \in Y)]\!]$.

(Collection): Given $x \in V^{\mathcal{P}}$, let

$$F_x = \{s \mid s \text{ is a fact in } \mathcal{P} \text{ and } \exists y \in V^{\mathcal{P}}([\![\varphi(x,y)]\!] = s)\}$$

Then F_x is a set and $\forall s \in F_x \exists\alpha\exists y([\![\varphi(x,y)]\!] = s \ and \ \rho(y) = \alpha)$. Hence by the Collection principle in ZF,

$$\exists v\forall s \in F_x\exists\alpha \in v\exists y([\![\varphi(x,y)]\!] = s \text{ and } \rho(y) = \alpha)$$

Let $\alpha_x = \bigcup\{\alpha \mid \alpha \in v \ and \ \alpha \in Ord\}$. Then

$$\forall s \in F_x\exists\alpha \in \alpha_x\exists y([\![\varphi(x,y)]\!] = s \text{ and } \rho(y) = \alpha)$$

That is to say,

$$F_x = \{s \mid s \text{ is a fact in } \mathcal{P} \text{ and } \exists y \in V^{\mathcal{P}}_{\alpha_x}([\![\varphi(x,y)]\!] = s)\}$$

Hence $\bigcup_{y\in V^{\mathcal{P}}}[\![\varphi(x,y)]\!] = \bigcup_{y\in V^{\mathcal{P}}_{\alpha_x}}[\![\varphi(x,y)]\!]$. We let $\beta = \bigcup\{\alpha_x \mid x \in dom(u)\}$. Then

$$\begin{aligned}
[\![\forall x \in u\exists y\varphi(x,y)]\!] &= \bigcap_{x\in dom(u)} (u(x) \multimap (\bigcup_{y\in V^{\mathcal{P}}}[\![\varphi(x,y)]\!])^{\perp\perp}) \\
&= \bigcap_{x\in dom(u)} (u(x) \multimap (\bigcup_{y\in V^{\mathcal{P}}_{\alpha_x}}[\![\varphi(x,y)]\!])^{\perp\perp}) \\
&\subset \bigcap_{x\in dom(u)} (u(x) \multimap (\bigcup_{y\in V^{\mathcal{P}}_{\beta}}[\![\varphi(x,y)]\!])^{\perp\perp})
\end{aligned}$$

Now let $v \in V^{\mathcal{P}}$ be such that $dom(v) = V^{\mathcal{P}}_{\beta}$ and $1 \in v(t)$ for all $t \in dom(v)$. Then

$$\begin{aligned}
(\bigcup_{y\in V^{\mathcal{P}}_{\beta}}[\![\varphi(x,y)]\!])^{\perp\perp} &\subset (\bigcup_{y\in dom(v)}(v(y) \otimes [\![\varphi(x,y)]\!]))^{\perp\perp} \\
&= [\![\exists y \in v(\varphi(x,y))]\!]
\end{aligned}$$

Hence

$$\bigcap_{x\in dom(u)} (u(x) \multimap (\bigcup_{y\in V^{\mathcal{P}}_{\beta}} [\![\varphi(x,y)]\!])^{\perp\perp}) \subset \bigcap_{x\in dom(u)} (u(x) \multimap [\![\exists y \in v(\varphi(x,y))]\!]).$$

(Infinity): We denote the phase-valued set obtained in **(Empty Set)** by $\emptyset_{\mathcal{P}}$. Similarly $\{x,\{x\}\}$ and $x\cup\{x\}$ denote the phase-valued sets obtained by **(Pair)** and **(Union)** for now. Define $Y \in V^{\mathcal{P}}$ in such a way that

- $\emptyset_{\mathcal{P}} \in dom(Y)$
- If $x \in dom(Y)$, then $x\cup\{x\} \in dom(Y)$
- $1 \in Y(\emptyset_{\mathcal{P}})$
- $Y(x) \subset Y(x\cup\{x\})$ for all $x \in dom(Y)$

Notice that if $\rho(x) = \alpha$, then $\rho(\{x\}) = \alpha+1$ and $\rho(\{x,\{x\}\}) = \alpha+2$. Also, $\rho(x\cup\{x\}) = \rho(\bigcup\{x,\{x\}\}) \le \rho(\{x,\{x\}\})$. Hence, given $\emptyset_{\mathcal{P}} \in V^{\mathcal{P}}_{\alpha}$, we can have $Y \in V^{\mathcal{P}}_{\alpha+\omega}$. Therefore, $Y \in V^{\mathcal{P}}$.

Since $1 \in Y(\emptyset_{\mathcal{P}}) \subset [\![\emptyset_{\mathcal{P}} \in Y]\!]$, it suffices to show

$$\begin{aligned} 1 \;&\in\; [\![\forall x(x\in Y \multimap x\cup\{x\} \in Y)]\!] \\ &= \bigcap_{x\in dom(Y)} (Y(x) \multimap (\bigcup_{z\in dom(Y)} (Y(z)\otimes [\![z = x\cup\{x\}]\!]))^{\perp\perp}) \end{aligned}$$

Now for any $x \in dom(Y)$,

$$\begin{aligned} Y(x) \;&\subset\; Y(x\cup\{x\}) \\ &\subset\; Y(x\cup\{x\}) \otimes [\![x\cup\{x\} = x\cup\{x\}]\!] \\ &\subset \bigcup_{z\in dom(Y)} (Y(z)\otimes [\![z = x\cup\{x\}]\!]) \\ &\subset (\bigcup_{z\in dom(Y)} (Y(z)\otimes [\![z = x\cup\{x\}]\!]))^{\perp\perp} \end{aligned}$$

□

4 Relating to the Heyting-valued models

Let's begin with the following observation, which is what is behind the Girard's second translation [3] of intuitionistic predicate logic into linear predicate logic:

$$\begin{array}{ll} A^* = {!}\,A \text{ for } A \text{ atomic} & \\ (A\wedge B)^* = A^*\otimes B^* & (A\vee B)^* = A^*\oplus B^* \\ (A\supset B)^* = {!}\,(A^*\multimap B^*) & \mathbf{0}^* = \mathbf{0} \\ (\forall xA)^* = {!}\,\forall x\, A^* & (\exists xA)^* = \exists x\, A^* \end{array}$$

Proposition 4.1. *Let $\mathcal{O}$ be the set of all open facts in $\mathcal{P}$. Then $\mathcal{O}$ is a locale.*

Proof. The order is given by the set-inclusion. The arbitrary join $\bigvee F_i$ is defined as $(\bigcup F_i)^{\perp\perp}$ which is open by the proposition 2.19 and the definition of closed facts. The binary meet of F and G is $F \otimes G$. This is confirmed by $F \otimes G = {!}\,F \otimes {!}\,G = {!}\,(F \cap G)$. Furthermore the arbitrary join and binary meet commute by the proposition 2.17. $\square$

Then we can obtain the Heyting-valued universe $V^{\mathcal{O}}$ as the subuniverse by restricting the truth values to open facts in the construction of $V^{\mathcal{P}}$ as follows:

Definition 4.2. *We define $V_\alpha^{\mathcal{O}}$ and $V^{\mathcal{O}}$ by the transfinite induction on ordinals α as follows:*

- $V_0^{\mathcal{O}} = \emptyset$
- $V_{\alpha+1}^{\mathcal{O}} = \{u \mid u$ *is a function with* $dom(u) \subset V_a^{\mathcal{O}}$ *and* $ran(u) = \mathcal{O}\}$
- $V_\lambda^{\mathcal{O}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathcal{O}}$ *where λ is a limit ordinal.*
- $V^{\mathcal{O}} = \bigcup_{\alpha \in ON} V_\alpha^{\mathcal{O}}$.

Definition 4.3. *A phase-valued set $u \in V^{\mathcal{P}}$ is static if $u \in V^{\mathcal{O}}$.*

Proposition 4.4. *Let $u \in V^{\mathcal{P}}$ be static. Then $[\![x \in u]\!]$ is open for all $x \in V^{\mathcal{P}}$.*

Proof. $[\![x \in u]\!] = (\bigcup_{z \in dom(u)}(u(z) \otimes [\![x = z]\!]))^{\perp\perp}$. Since open facts are closed under finite multiplicative conjunction, $[\![x = z]\!]$ is open. Furthermore, open facts are closed under infinitary additive disjunction. Hence, $[\![x \in u]\!]$ is open. $\square$

We introduce the restricted quantifications over $V^{\mathcal{O}}$:

- $[\![\exists v^* \phi(v, u_1, \ldots, u_n)]\!] = \bigcup_{v \in V^{\mathcal{O}}} [\![\phi(v, u_1, \ldots, u_n)]\!]$
- $[\![\forall v^* \phi(v, u_1, \ldots, u_n)]\!] = \bigcap_{v \in V^{\mathcal{O}}} [\![\phi(v, u_1, \ldots, u_n)]\!]$

The proposition 3.8 holds with those restricted quantifiers as well. The proofs are exactly the same.

For the counterparts of the power set axiom and H. Friedman's ϵ-induction [2, 5], it seems that we need to use those restricted quantifiers.

Theorem 4.5. *The following formulas are valid in $V^{\mathcal{P}}$.*

(Static Set): $\forall x^* \forall y\, (y \in x \multimap {!}\,(y \in x))$

(Static Power Set): $\forall u^* \exists v^* \forall x^* ({!}\,\forall y (y \in x \multimap y \in u) \multimap x \in v))$

(Static ϵ-induction): ${!}\,\forall x^*((\forall y\, (y \in x \multimap \phi(y)) \multimap \phi(x)) \multimap \forall x^* \phi(x)$

Proof.

(Static Set): This follows from the proposition 4.4.

(Static Power Set): Let $v \in V^{\mathcal{O}}$ be such that

$$dom(v) = \{f \mid f \text{ is static with } dom(f) = dom(u)\}$$

and $1 \in v(x)$ for all $x \in dom(v)$. We want to show $1 \in [\![\forall x^*(!x \subset u \multimap x \in v)]\!]$. For this, we define $x' \in dom(v)$ for each $x \in V^{\mathcal{O}}$ which satisfies:

$$[\![!\forall y(y \in x \multimap y \in u)]\!] = ![\![x \subset u]\!] \subset [\![x' = x]\!]$$

Given such an x', the validity of the formula immediately follows since $1 \in [\![x' \in v]\!]$ and $![\![x \subset u]\!] \subset [\![x' = x]\!] \subset [\![x' = x]\!] \otimes [\![x' \in v]\!] \subset [\![x \in v]\!]$ for all $x \in V^{\mathcal{O}}$.

The definition is as follows. Given $x \in V^{\mathcal{O}}$, let $x' \in V^{\mathcal{O}}$ be such that $dom(x') = dom(u)$ and $x'(y) = [\![y \in x]\!]$ for all $y \in dom(x')$. Clearly, $x' \in dom(v)$. Now for any $y \in V^{\mathcal{P}}$,

$$\begin{aligned}
[\![y \in x']\!] &= (\bigcup_{z \in dom(u)} x'(z) \otimes [\![z = y]\!])^{\perp\perp} \\
&= (\bigcup_{z \in dom(u)} [\![z \in x]\!] \otimes [\![z = y]\!])^{\perp\perp} \\
&\subset [\![y \in x]\!]
\end{aligned}$$

Hence $1 \in [\![\forall y(y \in x' \subset y \in x)]\!]$. Next for any $y \in V^{\mathcal{P}}$,

$$\begin{aligned}
[\![y \in u \otimes y \in x]\!] &= (\bigcup_{z \in dom(u)} (u(z) \otimes [\![z = y]\!] \otimes [\![y \in x]\!]))^{\perp\perp} \\
&\subset (\bigcup_{z \in dom(u)} ([\![z = y]\!] \otimes [\![z \in x]\!]))^{\perp\perp} \text{ since } u(z) \text{ is open} \\
&= (\bigcup_{z \in dom(u)} ([\![z = y]\!] \otimes x'(z)))^{\perp\perp} \\
&= [\![y \in x']\!]
\end{aligned}$$

Then for any $y \in V^{\mathcal{P}}$,

$$\begin{aligned}
[\![y \in x]\!] \otimes [\![\forall y(y \in x \multimap y \in u)]\!] &\subset [\![y \in x]\!] \otimes [\![y \in x]\!] \otimes [\![x \subset u]\!] \text{ since } x \in V^{\mathcal{O}} \\
&\subset [\![y \in x]\!] \otimes [\![y \in u]\!] \\
&\subset [\![y \in x']\!]
\end{aligned}$$

Hence $[\![\forall y(y \in x \multimap y \in u)]\!] \subset [\![\forall y(y \in x \multimap y \in x')]\!]$. Then we have

$$\begin{aligned}
[\![!\forall y(y \in x \multimap y \in u)]\!] &\subset [\![!\forall y(y \in x \multimap y \in x')]\!] \\
&\subset ![\![x \subset x']\!] \otimes ![\![x' \subset x]\!] \\
&= [\![x = x']\!]
\end{aligned}$$

(Static ϵ-induction): We show by the transfinite induction on the rank of $u \in V^{\mathcal{O}}$ that

$$[\![! \, \forall x^* ((\forall y \, (y \in x \multimap \phi(y^*)) \multimap \phi(x))]\!] \subset [\![\phi(u)]\!]$$

Note that we have

$$[\![! \, \forall x^* ((\forall y \, (y \in x \multimap \phi(y)) \multimap \phi(x))]\!] \subset (\bigcap_{y \in dom(u)} (u(y) \multimap [\![\phi(y)]\!])) \multimap [\![\phi(u)]\!]$$

For any $y \in dom(u) \subset V^{\mathcal{O}}$, the inductive hypothesis yields

$$\begin{aligned} [\![! \, \forall x^* ((\forall y \, (y \in x \multimap \phi(y)) \multimap \phi(x))]\!] & \subset [\![\phi(y)]\!] \\ & \subset u(y) \multimap [\![\phi(y)]\!] \end{aligned}$$

since u is static and $u(y)$ is open. Hence

$$[\![! \, \forall x^* ((\forall y \, (y \in x \multimap \phi(y)) \multimap \phi(x))]\!] \subset \bigcap_{y \in dom(u)} (u(y) \multimap [\![\phi(y)]\!])$$

The conclusion follows since $[\![! \, \forall x^* ((\forall y \, (y \in x \multimap \phi(y)) \multimap \phi(x))]\!]$ is open.

□

They are special consequences of the more general principle.

Proposition 4.6. *For the static u and v, our definition of $[\![u \in v]\!]$ and $[\![u = v]\!]$ in $V^{\mathcal{P}}$ yield the same open facts as the Heyting-valued interpretations in $V^{\mathcal{O}}$.*

Proof. Since the meet in $\mathcal{O}$ is the tensor in $\mathcal{P}$ and the *supremum* coincides in both of them, it suffices to confirm that $![\![u \subset v]\!]$ in the phase-valued model is the same as $[\![u \subset v]\!]$ in the Heyting-valued model for $u, v \in V^{\mathcal{O}}$.

Note that the *infimum* of open facts F_i in $\mathcal{O}$ is given by $! \bigcap F_i$. Furthermore $! \bigcap F_i = ! \bigcap ! F_i$ holds. Hence we only need to show that $!(u(x) \multimap [\![x \in u]\!])$ in $\mathcal{P}$ is indeed $u(x) \rightarrow [\![x \in u]\!]$ in $\mathcal{O}$.

Now $F \otimes !(F \multimap G) \subset G$ holds for any facts F and G. Suppose $F \otimes H \subset G$ for open facts F, G and H. Then $H \subset F \multimap G$ and $H \subset !(F \multimap G)$ since H is open. By the uniqueness of $F \rightarrow G$, we can conclude that $!(F \multimap G) = (F \rightarrow G)$.
□

Then the formulas in the intuitionistic set theory evaluated in $V^{\mathcal{O}}$ retain the same interpretations under the Girard's second translation with all the quantifiers modified to the restricted ones. Furthermore, if the quantifiers are bounded, then there is no need to restrict them due to the proposition 3.8. We hope to explore this point in more detail in the sequel of this paper.

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