A coherence space semantics for linear set theory

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Abstract

In this paper, we give a model for a naive set theory based on the MALL fragment of linear logic, using the coherence space semantics and the Scott-style style inverse limit construction. The main idea is to introduce an ordering in the set M of coherence spaces, with respect to which M becomes a cpo and all the logical operations are continuous. We are then able to construct the universe of M-valued sets by solving a certain domain equation.

1 Introduction

In this paper, we study the semantics of a naive set theory based on the multiplicative and additive fragment of linear logic. One of the reasons to consider such a system is that the set theoretical paradoxes do not hold in the absence of contraction. This phenomenon was known to early combinatory logicians such as Curry and Fitch in the 1930's [3], and Grishin proved the consistency of the naive set theory in affine logic in 1974 [8, 9]. Later, similar systems have been studied by White [19, 20] and Komori [12], and the author formulated the system LZF in full linear logic, which was proved to be a conservative extension of the standard Zermelo-Fraenkel set theory in classical logic [15, 16]. Recently, Girard considered a naive set theory in the framework of light linear logic [6].

The above mentioned works are, however, mostly syntactic. In fact, such a set theory behaves really well in terms of proof theory. For example, the cut-elimination or normalization for a system without the exponentials can be proved by the induction on ω, which is in sharp contrast with the classical or intuitionistic set theory [11]. The system can be always conservatively extended with fixpoints [6]. Furthermore, one can explicitly construct fixpoints and show that all the totally recursive functions are numeralwise representable, within the system with reasonable equality and paring [17].

On the other hand, the semantics for such a system has not been sufficiently developed. Komori gave a model of type-free combinatory logic in terms of Kripke semantics of affine logic [12, 13], but it is very difficult to construct except as the term model. The author studied a phase-space valued model, which is in analogue to Boolean-valued models and Heyting-Valued models, but it does not yield a model for a naive set theory [15]. The lack of a good semantics tends to make a system less convincing and appealing to many people. This paper tries to address the problem by constructing a reasonable model for a naive set theory based on linear logic.

Our strategy for the construction of the model is as follows. As it is always the case with the set theory based on a non-standard logic, sets are interpreted as functions from sets to truth values. In Boolean-valued models and Heyting-valued models, however, such functions are constructed step by step so that the the domain of a function of level $\alpha + 1$ is the partial universe V_α of the sets up to the level α. In short, the entire universe forms a cumulative hierarchy. In our case, the domain of each such function needs to be the entire universe because of the principle of the unrestricted comprehension.

Let V be our universe and M the set of truth values. For any formula $A(x)$, one can always construct the term $\{x : A(x)\}$. Then, it is most desirable to interpret the term $\{x : A(x)\}$ as an element of $[\![\{x : A(x)\}]\!]_\eta$ of V, on the one hand, and as the function $a \mapsto [\![A(x)]\!]_{\eta[x \mapsto a]}$ from V to M, on the other, where $a \in V$ and η is an assignment. In other words, the universe V needs to be isomorphic to the function space $[V \to M]$.

The setting is all too familiar to anyone who knows the model theory of untyped λ-calculus, and one can expect to apply the Scott-style method to the construction of V [18]. For this to be worked out, however, the set M of truth values should be a cpo under a certain ordering and the function space $[V \to M]$ be the set of all continuous functions. Furthermore, the latter needs to be closed under the logical operations of linear logic so that one can interpret complex formulas inductively. In particular, we require that if the function $a \mapsto [\![A(x)]\!]_{\eta[x \mapsto a]}$ for $A(x)$ is continuous, so be the function $a \mapsto [\![A(x)^\perp]\!]_{\eta[x \mapsto a]}$ for the linear negation $A(x)^\perp$. This causes some problem with the choice of M and the ordering. For example, if we take M to be a quantale and use its native ordering, then M is certainly a cpo, since it is a complete lattice. The linear negation on M is, however, not continuous, since it is not monotone after all.

Hence it is necessary to find a good M and a good ordering on M, with respect to which M becomes a cpo and all the logical operations of linear logic are continuous functions on M. In this paper, we choose a set of coherence spaces as M and introduce the ordering by the subspace relation among them. We then solve the domain equation $V \cong [V \to M]$ by the Scott-style inverse limit construction to yield the universe V of a naive set theory based on MALL fragment of linear logic. The coherence spaces are invented by Berry [1] and used by Girard for the semantics of the second-order λ-calculus [4], and they are supposed to be the original source and semantics of linear logic [5, 7]. Hence, the author believes that our choice is legitimate enough to assure the reasonableness of our model.

2 Preliminaries

We briefly review the basics of coherence spaces and complete partial orders (cpo's). For more thorough exposition, we refer the reader to the textbooks [7] and [1, 10, 14].

Definition 2.1. *A* coherence space *is a set (of sets)* $\mathcal{A}$ *which satisfies:*

1. *the downward closure; if* $a \in \mathcal{A}$ *and* $a' \subseteq a$*, then* $a' \in \mathcal{A}$*,*
2. *the binary completeness; if* $S \in \mathcal{A}$ *and* $\forall a_1, a_2 \in S\ (a_1 \cup a_2 \in \mathcal{A})$*, then* $\bigcup S \in \mathcal{A}$*.*

The elements of $|\mathcal{A}| = \bigcup \mathcal{A}$ are called *tokens*. The coherence space $\mathcal{A}$ can be identified with the graph $(|\mathcal{A}|, \frown)$, where $\frown$ is a reflexive and symmetric relation. The latter is given by the *coherence relation modulo* $\mathcal{A}$:

$$\alpha \frown \alpha' \pmod{\mathcal{A}} \quad \text{iff} \quad \{\alpha, \alpha'\} \in \mathcal{A}.$$

On the other hand, the graph $(|\mathcal{A}|, \frown)$ defines the coherence space $\mathcal{A}$ as the set of its complete subgraphs. The coherence relation modulo $\mathcal{A}$ is often denoted $\alpha \frown_{\mathcal{A}} \alpha'$ as well.

In the category of coherence spaces, the standard morphisms are *stable* functions. We refer the reader to Girard's textbook [7] for their definition. Importantly, the stable functions F from $\mathcal{A}$ to $\mathcal{B}$ can be described in terms of their *traces* $Tr(F)$, which are the set of pairs (a, β) with finite $a \in \mathcal{A}$ and $\beta \in |\mathcal{B}|$ such that a is the minimal element satisfying $\beta \in F(a)$. The set of all such traces then becomes a coherence space.

We are, however, interested in a model of linear logic. The stable function F is *linear* if the first element of pairs (a, β) in its trace $Tr(F)$ is a singleton $\{\alpha\}$. One can then simply replace the singleton $\{\alpha\}$ by the element $\alpha \in |\mathcal{A}|$ and use the result, called the *linear traces* $TrlinF$, for describing the linear function F.

The set of all linear functions from $\mathcal{A}$ to $\mathcal{B}$ allows a particularly pleasant characterization. For the elements of $\mathcal{A}$, we define the *incoherence* relation $\alpha \asymp \alpha' \pmod{\mathcal{A}}$ by

$$\alpha \asymp \alpha' \pmod{\mathcal{A}} \quad \text{iff} \quad \neg(\alpha \frown_{\mathcal{A}} \alpha') \text{ or } \alpha = \alpha'$$

where the condition $\alpha = \alpha'$ assures the reflexivity.

Fact 2.2. *The set of linear traces of all linear functions from* $\mathcal{A}$ *to* $\mathcal{B}$ *is the coherence space* $\mathcal{A} \multimap \mathcal{B}$ *defined by*

1. *the set of tokens;* $|\mathcal{A} \multimap \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$*,*
2. *the coherence relation;* $(\alpha, \beta) \frown (\alpha', \beta') \pmod{\mathcal{A} \multimap \mathcal{B}}$ *iff*
   - *if* $\alpha \frown_{\mathcal{A}} \alpha'$ *then* $\beta \frown_{\mathcal{B}} \beta$*, and*
   - *if* $\beta \asymp_{\mathcal{B}} \beta'$ *then* $\alpha \asymp_{\mathcal{A}} \alpha'$*.*

The incoherence relation itself yields the linear negation $\mathcal{A}^\perp$ of a coherence space $\mathcal{A}$. Furthermore the tensor product $\mathcal{A} \otimes \mathcal{B}$ of two coherence spaces $\mathcal{A}$ and $\mathcal{B}$ can be defined pairwise.

Definition 2.3. *The linear negation $\mathcal{A}^\perp$ of $\mathcal{A}$ is the coherence space defined by*

1. $|\mathcal{A}^\perp| = |\mathcal{A}|$,
2. $\alpha \frown \alpha' \pmod{\mathcal{A}^\perp}$ *iff* $\alpha \asymp_{\mathcal{A}} \alpha'$.

Definition 2.4. *The tensor product $\mathcal{A} \otimes \mathcal{A}$ of $\mathcal{A}$ and $\mathcal{B}$ is the coherence space defined by*

1. $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$,
2. $(\alpha, \beta) \frown (\alpha', \beta') \pmod{\mathcal{A} \otimes \mathcal{B}}$ *iff* $\alpha \frown_{\mathcal{A}} \alpha'$ *and* $\beta \frown_{\mathcal{B}} \beta'$.

The category of coherence spaces and linear functions is a $\star$-autonomous category with the $(-)^\perp$ as the dualizer and the singleton coherence space as the tensor unit. In addition, the Cartesian products $\mathcal{A}_1 \& \mathcal{A}_2$ and coproducts $\mathcal{A} \oplus \mathcal{B}$ are given by $|\mathcal{A}_1 \& \mathcal{A}_2| = |\mathcal{A}_1 \oplus \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| = \{1\} \times |\mathcal{A}_1| \cup \{2\} \times |\mathcal{A}_2|$ and

- $(i, \alpha) \frown (i, \alpha') \pmod{\mathcal{A}_1 \& \mathcal{A}_2}$ and $\pmod{\mathcal{A}_1 \oplus \mathcal{A}_2}$ iff $\alpha \frown_{\mathcal{A}_i} \alpha'$ for $i = 1, 2$,
- $(1, \alpha) \frown (2, \beta') \pmod{\mathcal{A}_1 \& \mathcal{A}_2}$ and $(1, \alpha) \asymp (2, \beta) \pmod{\mathcal{A}_1 \oplus \mathcal{A}_2}$ for all $\alpha \in \mathcal{A}_1$ and $\beta \in \mathcal{A}_2$.

The de Morgan duality holds between the Cartesian products and coproducts, and they give the interpretations of the additive operations in linear logic.

Next let $D = (D, \sqsubseteq)$ be a partially ordered set. A subset $X \subseteq D$ is *directed* if X is non-empty and for any two elements x, y in X there exists another element $z \in X$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. The poset D is a *complete partial order* (cpo) if

- there is a least element $\perp \in D$, and
- for every directed subset $X \subseteq D$, the supremum $\bigsqcup X$ exists.

In the category of cpo's, the morphisms are *continuous* functions which can be defined as

- the function $f : D \to D'$ is continuous iff $f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$ for all directed $X \subseteq D$.

This category is denoted **CPO**. The function space $[D \to D']$ is a cpo with the pointwise ordering and so is the cartesian product $D \times E$ with the pairwise ordering. Furthermore **CPO** is Cartesian closed.

3 The category Coh(T)

We use the coherence spaces as truth values of linear logic. The collection of all coherence spaces is, however, a proper class, which is not suitable for our construction. Hence we only consider the coherence spaces whose sets of tokens are subsets of a fixed non-empty set T. Furthermore we require T to be closed under pairing so that the set of all such coherence spaces is closed under the operations of linear logic. Note that T can be always constructed as the closure of an arbitrary non-empty set under the pairing operation.

Definition 3.1. *The* category of coherence spaces generated by T*, denoted* $\mathbf{Coh(T)}$*, consists of*

1. *the set $C(T)$ of all coherence spaces $\mathcal{A}$ with $|\mathcal{A}| \subseteq T$ as objects,*
2. *the set of all linear functions from $\mathcal{A}$ to $\mathcal{B}$ with $\mathcal{A}, \mathcal{B} \in C(T)$ as morphisms.*

Proposition 3.2. $\mathbf{Coh(T)}$ *is closed under the tensor product, linear negation and Cartesian product.*

We introduce a new ordering on the set $C(T)$ of coherence spaces by the *subspace* relation, under which $C(T)$ becomes a cpo.

Definition 3.3. *The coherence space $\mathcal{A} = (|\mathcal{A}|, \asymp_{\mathcal{A}})$ is a subspace of another coherence space $\mathcal{B} = (|\mathcal{B}|, \asymp_{\mathcal{B}})$ if*

1. *$|\mathcal{A}| \subseteq |\mathcal{B}|$, and*
2. *$\asymp_{\mathcal{A}} = \asymp_{\mathcal{B}\restriction\mathcal{A}}$,* i.e. *$\asymp_{\mathcal{A}}$ is the restriction of $\asymp_{\mathcal{B}}$ with respect to $|\mathcal{A}|$.*

This relation is denoted $\mathcal{A} \sqsubseteq \mathcal{B}$.

It can be easily checked that $\sqsubseteq$ is a partial order on $C(T)$, Furthermore $\emptyset \in C(T)$ is the bottom. In fact, $C(T)$ is a cpo under this ordering.

Lemma 3.4. $(C(T), \sqsubseteq)$ *is a cpo.*

Proof. Let $\mathcal{S} \subseteq C(T)$ be directed. Define $\bigsqcup \mathcal{S} = (|\bigsqcup \mathcal{S}|, \asymp)$ by $|\bigsqcup \mathcal{S}| = \bigcup_{\mathcal{A}\in\mathcal{S}} |\mathcal{A}|$ and $\asymp = \bigcup_{\mathcal{A}\in\mathcal{S}} \asymp_{\mathcal{A}}$. Each $\asymp_{\mathcal{A}}$ is reflexive and symmetric and so is $\asymp$. Hence $\bigsqcup \mathcal{S}$ is a coherence space.

$\bigsqcup \mathcal{S}$ is an upper bound of $\mathcal{S}$. Let $\mathcal{A} \in \mathcal{S}$. Then $|\mathcal{A}| \subseteq |\bigsqcup \mathcal{S}|$ and $\asymp_{\mathcal{A}} \subseteq \asymp$. Suppose $\alpha \asymp \alpha'$ and $\alpha, \alpha' \in \mathcal{A}$. Then $\alpha \asymp_{\mathcal{B}} \alpha'$ for some $\mathcal{B} \in \mathcal{S}$. Since $\mathcal{S}$ is directed, one can find $\mathcal{C}$ such that $\mathcal{A} \sqsubseteq \mathcal{C}$ and $\mathcal{B} \sqsubseteq \mathcal{C}$. Then $\alpha \asymp_{\mathcal{C}} \alpha'$ and $\alpha, \alpha' \in |\mathcal{A}|$, *i.e.* $\alpha \asymp_{\mathcal{C}\restriction\mathcal{A}} \alpha'$. Hence $\alpha \asymp_{\mathcal{A}} \alpha'$.

Furthermore $\bigsqcup \mathcal{S}$ is the least upper bound. Suppose $\mathcal{A} \sqsubseteq \mathcal{C}$ for all $\mathcal{A} \in \mathcal{S}$. Then $|\bigsqcup \mathcal{S}| = \bigcup |\mathcal{A}| \subseteq |\mathcal{C}|$. Similarly $\asymp = \bigcup \asymp_{\mathcal{A}} \subseteq \asymp_{\mathcal{C}}$. Let $\alpha \asymp_{\mathcal{C}} \alpha'$ and $\alpha, \alpha' \in |\bigsqcup \mathcal{S}|$. Then $\alpha \in \mathcal{A}_1$ and $\alpha' \in \mathcal{A}_2$ for some $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}$. Let $\mathcal{B} \in \mathcal{S}$ be such that $\mathcal{A}_1 \sqsubseteq \mathcal{B}$ and $\mathcal{A}_2 \sqsubseteq \mathcal{B}$. Then $\alpha \asymp_{\mathcal{C}\restriction\mathcal{B}} \alpha'$, and $\alpha \asymp_{\mathcal{B}} \alpha'$. Hence $\alpha \asymp \alpha'$. ☐

The linear negation, tensor product and Cartesian product can be regarded as the operations on this cpo. In addition they are continuous.

Proposition 3.5. *The operation of linear negation $\mathcal{A} \mapsto \mathcal{A}^\perp$ is monotone.*

Proof. Suppose $\mathcal{A} \sqsubseteq \mathcal{B}$. Then $|\mathcal{A}^\perp| = |\mathcal{A}| \subseteq |\mathcal{B}| = |\mathcal{B}^\perp|$. Let $\alpha \smile_{\mathcal{A}^\perp} \alpha'$. Then $\alpha \asymp_{\mathcal{A}} \alpha'$. Note that if $\alpha = \alpha'$, then $\alpha \smile_{\mathcal{B}^\perp} \alpha'$ trivially holds because of the reflexivity. Hence we may assume $\alpha \neq \alpha'$. Then $\neg(\alpha \smile_{\mathcal{A}} \alpha')$ and $\alpha, \alpha' \in |\mathcal{A}|$. If $\alpha \smile_{\mathcal{B}} \alpha'$, then $\alpha \smile_{\mathcal{B}^-\mathcal{A}} \alpha'$ *i.e.* $\alpha \smile_{\mathcal{A}} \alpha'$, This is a contradiction. Hence $\neg(\alpha \smile_{\mathcal{B}} \alpha')$ and $\alpha, \alpha' \in |\mathcal{B}|$. Therefore $\alpha \asymp_{\mathcal{B}} \alpha'$, *i.e.* $\alpha \smile_{\mathcal{B}^\perp} \alpha'$. On the other hand, let $\alpha \smile_{\mathcal{B}^\perp} \alpha'$ and $\alpha, \alpha' \in |\mathcal{A}^\perp|$. We may assume $\alpha \neq \alpha'$. Then $\neg(\alpha \smile_{\mathcal{B}} \alpha')$. Hence $\neg(\alpha \smile_{\mathcal{A}} \alpha')$. Therefore $\alpha \asymp_{\mathcal{A}} \alpha'$, *i.e.* $\alpha \smile_{\mathcal{A}^\perp} \alpha'$. $\square$

Lemma 3.6. *The operation of linear negation $\mathcal{A} \mapsto \mathcal{A}^\perp$ is continuous.*

Proof. Let $\mathcal{S} \subseteq C(T)$ be directed. Then $\{\mathcal{A}^\perp : \mathcal{A} \in \mathcal{S}\}$ is also directed and $\bigsqcup_{\mathcal{A}\in\mathcal{S}} \mathcal{A}^\perp = \bigcup_{\mathcal{A}\in\mathcal{S}} |\mathcal{A}| = |\bigsqcup \mathcal{S}| = |(\bigsqcup \mathcal{S})^\perp|$. Let $\alpha \smile_{\quad \mathcal{A}^\perp} \alpha'$. Then $\alpha, \alpha' \in |\mathcal{A}|$ and $\alpha \smile_{\mathcal{A}^\perp} \alpha'$ for some $\mathcal{A} \in \mathcal{S}$. We may assume $\alpha \neq \alpha'$. Then $\neg(\alpha \smile_{\mathcal{A}} \alpha')$. Suppose $\alpha \smile_{\mathcal{B}} \alpha'$ for some $\mathcal{B} \in \mathcal{S}$. Then one can find $\mathcal{C} \in \mathcal{S}$ such that $\mathcal{A} \sqsubseteq \mathcal{C}$ and $\mathcal{B} \sqsubseteq \mathcal{C}$. Then $\alpha \smile_{\mathcal{C}} \alpha'$ and $\alpha \smile_{\mathcal{C}^-\mathcal{A}} \alpha'$, *i.e.* $\alpha \smile_{\mathcal{A}} \alpha'$. This is a contradiction. Hence $\neg(\alpha \smile_{\mathcal{B}} \alpha')$ for all $\mathcal{B} \in \mathcal{S}$. In other words, $\neg(\alpha \smile_{\quad \mathcal{S}} \alpha')$ *i.e.* $\alpha \smile_{(\quad \mathcal{S})^\perp} \alpha'$. On the other hand, let $\alpha \smile_{(\quad \mathcal{S})^\perp} \alpha'$. We may assume $\alpha \neq \alpha'$. Then $\alpha, \alpha' \in \mathcal{A}$ for some $\mathcal{A} \in \mathcal{S}$ and $\neg(\alpha \smile_{\mathcal{B}} \alpha')$ for all $\mathcal{B} \in \mathcal{S}$. Hence $\alpha \smile_{\mathcal{A}^\perp} \alpha'$ and $\alpha \smile_{\quad \mathcal{A}^\perp} \alpha'$. $\square$

Proposition 3.7. *The operation of tensor product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \otimes \mathcal{B}$ is a monotone function from $C(T) \times C(T)$ to $C(T)$.*

Proof. Let $(\mathcal{A}, \mathcal{B}) \sqsubseteq_{C(T)\times C(T)} (\mathcal{A}', \mathcal{B}')$. Then $\mathcal{A} \sqsubseteq \mathcal{A}'$ and $\mathcal{B} \sqsubseteq \mathcal{B}'$. Hence $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}| \subseteq |\mathcal{A}'| \times |\mathcal{B}'| = |\mathcal{A}' \otimes \mathcal{B}'|$. Let $(\alpha, \beta) \smile_{\mathcal{A}\otimes\mathcal{B}} (\alpha', \beta')$. Then $\alpha \smile_{\mathcal{A}} \alpha'$ and $\beta \smile_{\mathcal{B}} \beta'$. Hence $\alpha \smile_{\mathcal{A}'} \alpha'$ and $\beta \smile_{\mathcal{B}'} \beta'$, *i.e.* $(\alpha, \beta) \smile_{\mathcal{A}'\otimes\mathcal{B}'} (\alpha', \beta')$. On the other hand, let $(\alpha, \beta) \smile_{\mathcal{A}'\otimes\mathcal{B}'} (\alpha', \beta')$ and $(\alpha, \beta), (\alpha', \beta') \in |\mathcal{A} \otimes \mathcal{B}|$. Then $\alpha \smile_{\mathcal{A}'^-\mathcal{A}} \alpha'$ and $\beta \smile_{\mathcal{B}'^-\mathcal{B}} \beta'$. Hence $\alpha \smile_{\mathcal{A}} \alpha'$ and $\beta \smile_{\mathcal{B}} \beta'$, *i.e.* $(\alpha, \beta) \smile_{\mathcal{A}\otimes\mathcal{B}} (\alpha', \beta')$ $\square$

Lemma 3.8. *The operation of tensor product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \otimes \mathcal{B}$ is a continuous function from $C(T) \times C(T)$ to $C(T)$.*

Proof. Let $\mathcal{S} \subseteq C(T) \times C(T)$ be directed. Then $\{\mathcal{A} \otimes \mathcal{B} : (\mathcal{A}, \mathcal{B}) \in \mathcal{S}\}$ is also directed. Let $\mathcal{S}_1 = \{\mathcal{A} : \exists\mathcal{Y}\ (\mathcal{A}, \mathcal{Y}) \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{\mathcal{B} : \exists\mathcal{X}\ (\mathcal{X}, \mathcal{B}) \in \mathcal{S}\}$. Then $\bigsqcup \mathcal{S} = (\bigsqcup \mathcal{S}_1, \bigsqcup \mathcal{S}_2)$ and $|\bigsqcup_{(\mathcal{A},\mathcal{B})\in\mathcal{S}} \mathcal{A} \otimes \mathcal{B}| = \bigcup_{(\mathcal{A},\mathcal{B})\in\mathcal{S}} |\mathcal{A}| \times |\mathcal{B}| \subseteq \bigcup_{\mathcal{A}\in\mathcal{S}_1} |\mathcal{A}| \times \bigcup_{\mathcal{B}\in\mathcal{S}_2} |\mathcal{B}| = |\bigsqcup \mathcal{S}_1 \otimes \bigsqcup \mathcal{S}_2|$. On the other hand, suppose $(\alpha, \beta) \in |\bigsqcup \mathcal{S}_1 \otimes \bigsqcup \mathcal{S}_2|$. Then $\alpha \in |\mathcal{C}|$ and $\beta \in |\mathcal{D}|$ for some $\mathcal{C} \in \mathcal{S}_1$ and $\mathcal{D} \in \mathcal{S}_2$. Since $\mathcal{S}$ is directed, one can find $(\mathcal{C}', \mathcal{D}') \in \mathcal{S}$ such that $\mathcal{C} \sqsubseteq \mathcal{C}'$ and $\mathcal{D} \sqsubseteq \mathcal{D}'$. Hence $(\alpha, \beta) \in |\mathcal{C}'| \times |\mathcal{D}'| = |\mathcal{C}' \otimes \mathcal{D}'| \subseteq |\bigsqcup_{(\mathcal{A},\mathcal{B})\in\mathcal{S}} \mathcal{A} \otimes \mathcal{B}|$.

Suppose $(\alpha, \beta) \smile_{\quad \mathcal{A}\otimes\mathcal{B}} (\alpha', \beta')$. Then $(\alpha, \beta) \smile_{\mathcal{C}\otimes\mathcal{D}} (\alpha', \beta')$ for some $(\mathcal{C}, \mathcal{D}) \in \mathcal{S}$. Hence $\alpha \smile_{\mathcal{C}} \alpha'$ and $\beta \smile_{\mathcal{D}} \beta'$. Therefore $\alpha \smile_{\quad \mathcal{S}_1} \alpha'$ and $\beta \smile_{\quad \mathcal{S}_2} \beta'$, *i.e.*

$$(\alpha, \beta) \smile_{\quad \mathcal{S}_1 \otimes \quad \mathcal{S}_2} (\alpha', \beta').$$

On the other hand, suppose $(\alpha, \beta) \smile_{\quad \mathcal{S}_1 \otimes \quad \mathcal{S}_2} (\alpha', \beta')$. Then $\alpha \smile_{\mathcal{C}} \alpha'$ and $\beta \smile_{\mathcal{D}} \beta'$ for some $\mathcal{C} \in \mathcal{S}_1$ and $\mathcal{D} \in \mathcal{S}_2$. Let $(\mathcal{C}', \mathcal{D}') \in \mathcal{S}$ be such that $\mathcal{C} \sqsubseteq \mathcal{C}'$ and $\mathcal{D} \sqsubseteq \mathcal{D}'$, Then $(\alpha, \beta) \smile_{\mathcal{C}'\otimes\mathcal{D}'} (\alpha', \beta')$, Hence $(\alpha, \beta) \smile_{\quad \mathcal{A}\otimes\mathcal{B}} (\alpha', \beta')$. $\square$

Since the par $\mathcal{A}⅋\mathcal{B}$ can be defined by the de Morgan duality as $(\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$, all the multiplicative operations of linear logic are continuous with respect to our ordering. Besides, the additive operations are continuous as well. By the de Morgan duality, it suffices to confirm this for the Cartesian product.

Proposition 3.9. *The operation of Cartesian product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A}\&\mathcal{B}$ is a monotone function from $C(T) \times C(T)$ to $C(T)$.*

Proof. Let $(\mathcal{A}, \mathcal{B}) \sqsubseteq_{C(T)\times C(T)} (\mathcal{A}', \mathcal{B}')$. Then $\mathcal{A} \sqsubseteq \mathcal{A}'$ and $\mathcal{B} \sqsubseteq \mathcal{B}'$. Hence $|\mathcal{A}\&\mathcal{B}| = |\mathcal{A}| + |\mathcal{B}| \subseteq |\mathcal{A}'| + |\mathcal{B}'| = |\mathcal{A}'\&\mathcal{B}'|$. For $\alpha \in |\mathcal{A}|$ and $\beta \in |\mathcal{B}|$, the injections $(1, \alpha)$ and $(2, \beta)$ are always related in both $\mathcal{A}\&\mathcal{B}$ and $\mathcal{A}'\&\mathcal{B}'$. Furthermore

$$\begin{aligned} (1,\alpha)\smile_{\mathcal{A}\&\mathcal{B}}(1,\alpha') &\text{ iff } \alpha\smile_{\mathcal{A}}\alpha' \\ &\text{ iff } \alpha\smile_{\mathcal{A}'^{-}\mathcal{A}}\alpha' \\ &\text{ iff } (1,\alpha)\smile_{\mathcal{A}'\&\mathcal{B}'^{-}\mathcal{A}\&\mathcal{B}}(1,\alpha') \end{aligned}$$

and similarly for $(2, \beta)$ and $(2, \beta')$. ☐

Lemma 3.10. *The operation of Cartesian product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A}\&\mathcal{B}$ is a continuous function from $C(T) \times C(T)$ to $C(T)$,*

Proof. Let $\mathcal{S} \subseteq C(T)$ be directed. Then $\{\mathcal{A}\&\mathcal{B} : (\mathcal{A}, \mathcal{B}) \in \mathcal{S}\}$ is also directed. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be defined as before. Then $|\bigsqcup \mathcal{S}_1 \& \bigsqcup \mathcal{S}_2| = \bigcup_{\mathcal{A}\in\mathcal{S}_1} |\mathcal{A}| + \bigcup_{\mathcal{B}\in\mathcal{S}_2} |\mathcal{B}| = \bigcup_{(\mathcal{A},\mathcal{B})\in\mathcal{S}} |\mathcal{A}| + |\mathcal{B}| = |\bigsqcup_{(\mathcal{A},\mathcal{B})\in\mathcal{S}} \mathcal{A}\&\mathcal{B}|$.

Let $\alpha \in |\mathcal{S}_1|$ and $\beta \in |\mathcal{S}_2|$. Then $\alpha \in |\mathcal{C}|$ and $\beta \in |\mathcal{D}|$ for some $\mathcal{C} \in \mathcal{S}_1$ and $\mathcal{D} \in \mathcal{S}_2$. Let $(\mathcal{C}', \mathcal{D}') \in \mathcal{S}$ such that $\mathcal{C} \sqsubseteq \mathcal{C}'$ and $\mathcal{D} \sqsubseteq \mathcal{D}'$. Then $(1, \alpha)\smile_{\mathcal{C}'\&\mathcal{D}'}(2, \beta)$, *i.e.* $(1, \alpha)\smile_{\quad\mathcal{A}\&\mathcal{B}}(2, \beta)$. Clearly $(1, \alpha)\smile_{\quad\mathcal{S}_1\&\quad\mathcal{S}_2}(2, \beta)$. Next let $\alpha, \alpha' \in |\mathcal{S}_1|$. Then

$$\begin{aligned} (1,\alpha)\smile_{\quad\mathcal{S}_1\&\quad\mathcal{S}_2}(1,\alpha') &\text{ iff } \alpha\smile_{\quad\mathcal{S}_1}\alpha' \\ &\text{ iff } \alpha\smile_{\mathcal{C}}\alpha' \text{ for some } \mathcal{C}\in\mathcal{S}_1 \\ &\text{ iff } (1,\alpha)\smile_{\mathcal{C}\&\mathcal{D}}(1,\alpha') \text{ for some } (\mathcal{C},\mathcal{D})\in\mathcal{S} \\ &\text{ iff } (1,\alpha)\smile_{\quad\mathcal{A}\&\mathcal{B}}(1,\alpha') \end{aligned}$$

and similarly for $\beta, \beta' \in |\mathcal{S}_2|$. ☐

Note that the ordering $\mathcal{A} \sqsubseteq \mathcal{B}$ naturally induces a linear function from $\mathcal{A}$ to $\mathcal{B}$ with the linear trace $\{(\alpha, \alpha) : \alpha \in |\mathcal{A}|\}$. This is the ordinary inclusion map of $\mathcal{A}$ into $\mathcal{B}$. The existence of inclusion map, however, does not necessarily yield $\mathcal{A} \sqsubseteq \mathcal{B}$ since $\alpha\smile_{\mathcal{B}}\beta$ may hold even when α is not related to β in $\mathcal{A}$.

4 The construction of the universe V

The universe V of $C(T)$-valued sets is constructed by the Scott style inverse limit construction as the fixpoint $V \cong [V \to C(T)]$. Since this is a simplified version of the well-known D^∞ construction, we only give a brief sketch of it.

The set $[D \to E]$ of all continuous functions from a cpo D to a cpo E is a cpo by the pointwise ordering, *i.e.*

$$f \sqsubseteq_{[D \to E]} g \text{ iff } f(x) \sqsubseteq_E g(x) \text{ for all } x \in D$$

for $f, g \in [D \to E]$. The pair of continuous functions $f : D \to E$ and $g : E \to D$ is called *an embedding-projection pair from D to E* iff $p \circ e = Id_D$ and $e \circ p \sqsubseteq_{E \to E} Id_E$. Our construction is carried out in the category $\mathbf{CPO_e}$ of cpo's and embedding-projection pairs. We define the operation F by

- $F(D) = [D \to C(T)]$,
- $F(e,p)$ is the embedding-projection pair of $f \mapsto f \circ p$ and $g \mapsto g \circ e$ with $f \in [D \to C(T)]$ and $g \in [E \to C(T)]$.

Then F is a covariant functor on this category.

Let $\top$ be the cpo which consists of a singleton set. Then $\top$ is an initial object of $\mathbf{CPO_e}$. In particular, there is a morphism (e_0, p_0) from $\top$ to $F(\top)$. Let $F^n(\top)$ and (e_n, p_n) be the results of F applied to $\top$ and (e_0, p_0) for n times, respectively. We then consider the diagram:

$$\top \xrightarrow{(e_0,p_0)} F^1(\top) \xrightarrow{(e_1,p_1)} F^2(\top) \xrightarrow{(e_2,p_2)} \cdots F^n(\top) \xrightarrow{(e_n,p_n)} F^{n+1}(\top) \cdots$$

Our fixpoint will be a colimit ΣF of this diagram. Let $\Pi_{n\in\omega} F^n(\top)$ be the set-theoretical product of $F^n(\top)$ and a be one of its elements. The n-th projection of a is simply denoted a_n. The object ΣF is then defined by

- $\Sigma F = \{a \in \Pi_{n\in\omega} F^n(\top) : a_n = p_n(a_{n+1}) \text{ for all } n \in \omega\}$
- $a \sqsubseteq_{\Sigma F} b$ iff $a_n \sqsubseteq_{F^n(\top)} b_n$ for all $n \in \omega$.

ΣF is a colimit of the diagram in $\mathbf{CPO_e}$ with the embedding-projection pairs (η_n, π_n) from $F^n(\top)$ to ΣF given by

- π_n is the set-theoretical projection, *i.e.* $\pi_n(a) = a_n$,
- $\eta_n(x)$ is the element $a \in \Sigma F$ such that $a_n = x$ and $a_{m+1} = e_m(a_m)$ for all $m \geq n$.

Furthermore the colimit ΣF is preserved under F, *i.e.* its image $F(\Sigma F)$ is a colimit of the diagram:

$$F^1(\top) \xrightarrow{(e_1,p_1)} F^2(\top) \xrightarrow{(e_2,p_2)} \cdots F^{n+1}(\top) \xrightarrow{(e_{n+1},p_{n+1})} F^{n+2}(\top) \cdots$$

Clearly ΣF itself is a colimit of this second diagram as well. Hence ΣF is isomorphic to $F(\Sigma F)$, *i.e.* the object ΣF is the required fixpoint $V \cong [V \to C(T)]$.

5 The interpretation of linear set terms in V

The naive set theory we consider is based on the multiplicative-additive fragment (MALL) of linear logic. It is enhanced with the set abstraction but without quantifiers. The terms and formulas are defined inductively:

1. the variables x, y, z, $\cdots$ are terms;
2. the constants $\mathbf{1}$, $\perp$, $\top$ and $\mathbf{0}$ are (atomic) formulas;
3. if s and t are terms, then $s \in t$ and $s \notin t$ are (atomic) formulas;
4. if A is a formula and v is a variable, then $\{v : A\}$ is a term;
5. if A and B be are formulas, so are $A \otimes B$, $A ⅋ B$, $A \& B$ and $A \oplus B$.

The free and bound variables in formulas are defined as usual. The linear negations $A^{\perp}$ of formulas A are given by the standard de Morgan duality in addition with

- $(s \in t)^{\perp} = s \notin t$ and $(s \notin t)^{\perp} = s \in t$.

Our inference system is the two-sided Gentzen-style sequent calculus for MALL enhanced with the new rules of inference for the set abstraction

$$\frac{\Gamma, A[s/v] \vdash \Delta}{\Gamma, s \in \{v : A\} \vdash \Delta} \quad \frac{\Gamma \vdash A[s/v], \Delta}{\Gamma \vdash s \in \{v : A\}, \Delta}$$

where $A[s/v]$ is the result of the substitution of the term s for the variable v in the formula A.

The terms s and formulas A with n free variables are interpreted by morphisms $V^n \overset{[\![s]\!]}{\to} V$ and $V^n \overset{[\![s]\!]}{\to} C(T)$ in the category **CPO**. Let ϕ be the isomorphism $V \overset{\phi}{\cong} [V \to C(T)]$ and $\hat{f}$ be the transpose of the morphism f. Furthermore we assume the alignment of the number of free variables by appropriate canonical morphisms. Then the interpretation can be assigned inductively as follows:

1. $[\![\mathbf{1}]\!]$ and $[\![\perp]\!]$ are singleton coherence spaces;
2. $[\![\top]\!]$ and $[\![\mathbf{0}]\!]$ are the empty coherence space;
3. $[\![v]\!]$ for the i-th variable v is the projection $V^n \overset{\pi_i}{\to} V$;
4. $[\![s \in t]\!]$ is the composition

$$V^n \xrightarrow{\langle [\![s]\!], [\![t]\!] \rangle} V \times V \xrightarrow{Id_{\times} \phi} V \times [V \to C(T)] \xrightarrow{eval} C(T);$$

5. $[\![\{v : A\}]\!]$ is the composition

$$V^{n-1} \xrightarrow{[\![A]\!]} [V \to C(T)] \xrightarrow{\phi^{-1}} V;$$

6. $[\![A^\perp]\!]$ is the composition

$$V^n \xrightarrow{[\![A]\!]} C(T) \xrightarrow{\perp} C(T)$$

where $\perp$ is the operation of linear negation;

7. $[\![A \otimes B]\!]$ is the composition

$$V^n \xrightarrow{\langle [\![A]\!],[\![B]\!] \rangle} C(T) \times C(T) \xrightarrow{\otimes} C(T)$$

where $\otimes$ is the operation of tensor product;

8. $[\![A \& B]\!]$ is the composition

$$V^n \xrightarrow{\langle [\![A]\!],[\![B]\!] \rangle} C(T) \times C(T) \xrightarrow{\&} C(T)$$

where $\&$ is the operation of Cartesian product.

9. $[\![A \parr B]\!]$ and $[\![A \oplus B]\!]$ are given by the de Morgan duality.

The morphisms $V^n \to C(T)$ in **CPO** are monotone and can be regarded as functors from the category V^n to the category $\mathbf{Coh(T)}$ with the ordering $\mathcal{A} \sqsubseteq_{C(T)} \mathcal{B}$ now read as the inclusion map. Then the sequents $\Gamma \vdash \Delta$ are interpreted as a natural transformation from $[\![\Gamma^\otimes]\!]$ to $[\![\Delta^\parr]\!]$ where $\Gamma^\otimes$ and $\Delta^\parr$ are the tensor and par products of all occurrences of formulas in Γ and Δ, respectively. Note that the interpretation of the formula $A[s/v]$ can be computed by the composition

$$V^n \xrightarrow{Id \times Id} V^n \times V^n \xrightarrow{Id \times [\![s]\!]} V^{n+1} \xrightarrow{[\![A]\!]} C(T)$$

and $[\![s \in \{v : A\}]\!] = [\![A[s/v]]\!]$ holds. Hence the inference rules for the set abstraction are sound as well as the axioms and other inference rules.

6 Conclusion

We constructed a model of a naive set theory based on MALL by combining the coherence space semantics for propositional linear logic and the Scott-style inverse limit construction. The main reason for this to be possible is that one can define the ordering among coherence spaces with respect to which the linear negation is a monotone, *i.e.* covariant, operation. This seems to show one of the special features of linear negation as opposed to intuitionistic or classical negation.

Our model is very natural and sufficiently model-theoretic. It is not, however, completely satisfactory. One of such unsatisfactory points is that our system of naive set theory does not have quantifiers although it has the set abstraction. The interpretation of a formula $A(x)$ with one free variable x is a map $[\![A(x)]\!] : V \to C(T)$ and the obvious candidate for the interpretation of $\forall x\, A(x)$ is the coherence space $\Pi_{a \in V} [\![A(X)]\!](a)$. In general, however, the latter does not reside in $C(T)$. For example, if T is a countable set, then $C(T)$ is uncountable and so is V. Then we do not have enough elements of T to use as indices for each $[\![A(x)]\!](a)$. The other unsatisfactory point is that the coherence space semantics is by no means complete with respect to propositional linear logic. In particular, the constants $\mathbf{1}$ and $\top$ are self-dual, *i.e.* $[\![\mathbf{1}]\!] = [\![\bot]\!]$ and $[\![\top]\!] = [\![\mathbf{0}]\!]$.

For further study, we need to address those issues. One direction seems to extend the truth-value set $C(T)$, on the one hand, and consider a structure more restrictive than cpo, on the other. Another direction is to use different types of semantics from coherence space semantics as the base model of propositional linear logic. For example, a certain version of game semantics seems promising.

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