Notes of *BROWNIAN MOTION (GPP)*

Yoann Potiron
Keio University

Any questions? Contact me by email to potiron@keio.jp

1 Technical tools

1.1 Probability space

We start by introducing the theoretical framework of probability theory. A probability space consists of three ingredients:

1. A set of all possible outcomes Ω.
2. A set of events
3. A probability function $\mathbb{P}$, which assigns to each event a number between 0 and 1.

For example, we can consider the event that the average temperature during a day is at least equal to 20 degrees. If we are in a summer day in Japan, this probability should be very close to 1. If we are in a winter day, this probability should be closer to 0.
For more advanced students: In general, there is no specification on Ω, so it is a "general" space, but it can happen that it corresponds to the six possible outcomes of a dice roll, etc.

1.2 Continuous random variable

A random variable is a function X from the sample space to the set of real numbers $\mathbb{R}$, where $\mathbb{R} := (-\infty, \infty)$. The probability that X is between a and b is defined as

$$\mathbb{P}(X \in [a,b]) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in [a,b]\}). \tag{1}$$

For example, X can be equal to the current price of Mitsubishi asset, the current temperature in Tokyo, the inflation rate in Japan, etc.
If X has a density f_X, the probability that X is between a and b is equal to

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(s)ds. \tag{2}$$

The density is positive and can be interpreted as the instantaneous probability. It corresponds to the area under the curve in Figure 1. We always have that the total area below the density is equal to 1. This can be expressed as

$$\int f_X(s)ds = 1. \tag{3}$$

This is natural since the total probability is always equal to 1.

The only density that we will be looking at in this class is the normal distribution, which we will be defining in a later section.
For more advanced students: The definition can be generalized in the two-dimensional case, where we consider the joint density. More specifically, if X and Y are two random variables, then their joint density takes on the form $f_{X,Y}(s, t)$.

1.3 Independence

Two random variables are independent if the realization of one does not affect the realization of the other. For example, if X stands for the current temperature in Tokyo, and Y corresponds to the number of students enrolled in this class, it would be reasonable to assume that X and Y are independent.

If we add a third random variable Z, which is the current temperature in Yokohama, then it would be reasonable to assume Y and Z are independent, but it would be unreasonable to assume independence between X and Z. Indeed, if it is cold in Tokyo, it will be most likely cold in Yokohama too, and vice versa.
For more advanced students: Two random variables X and Y are independent if their joint density and their own density satisfy

$$f_{X,Y}(s, t) = f_X(s)f_Y(t) \tag{4}$$

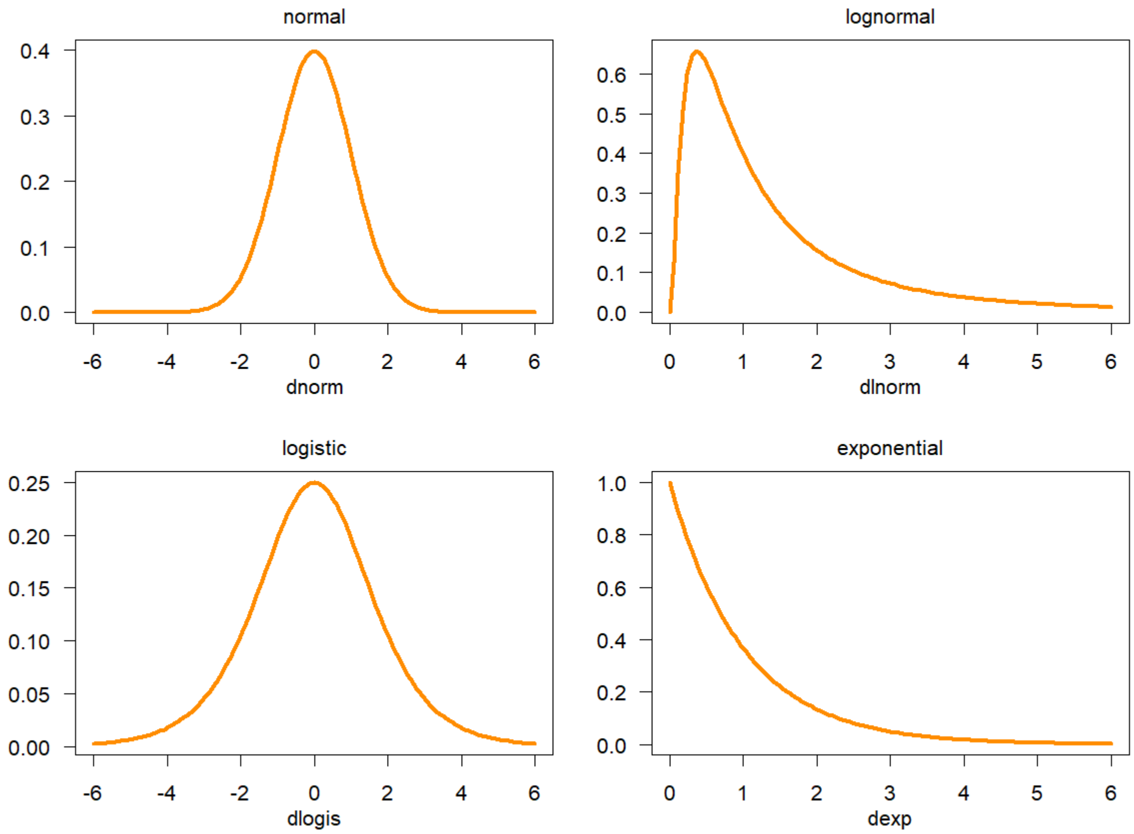

Figure 1: Four different densities

1.4 Calculation properties

First, we have that the sum of two random variables, when squared, is equal to the square of each random variable plus two times the product of each random variable, i.e.

$$(X+Y)^2 = X^2 + Y^2 + 2XY \tag{5}$$

We also denote the sum symbol as $\sum$, i.e.

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n \tag{6}$$

Then, the above property can be extended to the sum of n random variables squared, i.e.

$$\left(\sum_{i=1}^{n} X_i\right)^2 = \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right) \tag{7}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \tag{8}$$

$$= \sum_{i=1}^{n} X_i^2 + \sum_{1\leq i,j\leq n \text{ s.t. } i\neq j} X_i X_j \tag{9}$$

In the property (9), there is $n(n-1)$ terms in the sum $\sum_{1\leq i,j\leq n \text{ s.t. } i\neq j}$.

1.5 Mean

Now we define the mean as

$$\text{Mean}[X] = \int_{\mathbb{R}} s f_X(s) ds. \tag{10}$$

The mean is a number representing the "center" of a collection of numbers and is intermediate to the extreme values of the set of numbers. The mean is a NUMBER and not a random variable.

We provide now some very useful properties about the mean. First, we have that the mean of a number c is equal to the number c itself, i.e.,

$$\text{Mean}[c] = c. \tag{11}$$

Second, we have that the mean of a number times a random variable equals the number times the mean of the random variable, i.e.,

$$\text{Mean}[cX] = c\,\text{Mean}[X]. \tag{12}$$

Third, we have that the mean of the sum of two random variables is equal to the sum of the mean of each random variable, i.e.

$$\text{Mean}[X+Y] = \text{Mean}[X] + \text{Mean}[Y]. \tag{13}$$

This property also generalizes when there are more than two random variables, i.e.

$$\text{Mean}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \text{Mean}[X_i] \tag{14}$$

When X and Y are INDEPENDENT, we have that the mean of the product is equal to the product of the mean, i.e.

$$\text{Mean}[XY] = \text{Mean}[X]\,\text{Mean}[Y] \tag{15}$$

When $(X_1, \ldots, X_n)$ are INDEPENDENT, the property generalizes, i.e.

$$\text{Mean}\left[\Pi_{i=1}^{n} X_i\right] = \Pi_{i=1}^{n}\,\text{Mean}[X_i] \tag{16}$$

1.6 Variance

We define the variance as

$$\text{Var}[X] = \text{Mean}[(X - \text{Mean}[X])^2] = \int_{\mathbb{R}} (s - \text{Mean}[X])^2 f_X(s) ds. \tag{17}$$

The variance is a number representing how close the the random variable is from its mean. The variance is always positive. An important property of the variance is that the variance of a random variable is equal to the mean of the squared random variable minus the mean of the random variable, squared, i.e.

$$\text{Var}[X] = \text{Mean}[X^2] - \text{Mean}[X]^2. \tag{18}$$

We provide now some very useful properties about the variance. First, we have that the variance of a number c is equal to 0, i.e.

$$\text{Var}[c] = 0. \tag{19}$$

The reason is that a number is a random variable which is always equal to c. Second, we have that the variance of a number times a random variable equals the squared number times the mean of the random variable, i.e.

$$\text{Var}[cX] = c^2\,\text{Var}[X]. \tag{20}$$

When X and Y are INDEPENDENT, we have that the variance of the sum is equal to the sum of the variance, namely

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]. \tag{21}$$

When $(X_1, \ldots, X_n)$ are INDEPENDENT, the property generalizes, namely

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{Var}[X_i]. \tag{22}$$

1.7 Covariance

We define the covariance as

$$\begin{aligned} \mathrm{Cov}[X,Y] &= \mathrm{Mean}[(X-\mathrm{Mean}[X])(Y-\mathrm{Mean}[Y])] & (23) \\ &= \int_{\mathbb{R}\times\mathbb{R}} (s-\mathrm{Mean}[X])(t-\mathrm{Mean}[Y]) f_{X,Y}(s,t)\,dsdt. & (24) \end{aligned}$$

The covariance is a number representing the linear association between the two variables X and Y. The covariance can be positive or negative. If the covariance between two random variables is positive, it means that their linear association is in the same direction. If the covariance between two random variables is negative, it means that their linear association is in the opposite direction. If the covariance between two random variables is null, it means that there is no linear association between them.

An important property of the covariance is that the covariance of two random variables is equal to the mean of the product of both random variables minus the product of the mean of each random variable, i.e.

$$\mathrm{Cov}[X,Y] = \mathrm{Mean}[XY] - \mathrm{Mean}[X]\,\mathrm{Mean}[Y]. \tag{25}$$

In particular, this implies that the covariance between a number c and a random variable X is equal to 0, namely

$$\mathrm{Cov}[c,X] = 0. \tag{26}$$

The reason is that a number is a random variable which is always equal to itself. When X and Y are INDEPENDENT, we have that the covariance of X and Y is null, i.e.

$$\mathrm{Cov}[X,Y] = 0. \tag{27}$$

Indeed, there is no linear association between X and Y when they are independent. We have that the covariance between a random variable X and the sum of two random variables $Y + Z$ is equal to the sum of covariances, namely

$$\mathrm{Cov}[X, Y + Z] \quad = \quad \mathrm{Cov}[X, Y] + \mathrm{Cov}[X, Z]. \tag{28}$$

This property also generalizes when there are more than two random variables, i.e.

$$\mathrm{Cov}\left[X, \sum_{i=1}^{n} Y_i\right] \quad = \quad \sum_{i=1}^{n} \mathrm{Cov}[X, Y_i]. \tag{29}$$

The covariance property (25) is a generalization of the variance property (18). There are relations between the variance and the covariance. First, the definition (23) directly implies that the covariance between the same random variable is equal to its variance, i.e.

$$\mathrm{Cov}[X, X] \quad = \quad \mathrm{Var}[X]. \tag{30}$$

Second, the variance of the sum of two random variables is equal to the sum of each random variable variance plus two times the covariance, i.e.

$$\mathrm{Var}[X + Y] \quad = \quad \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]. \tag{31}$$

This property also generalizes when there are more than two random variables, i.e.

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}[X_i, X_j] \tag{32}$$

$$\quad = \quad \sum_{i=1}^{n} \mathrm{Var}[X_i] + \sum_{1 \le i,j \le n \text{ s.t. } i \ne j} \mathrm{Cov}[X_i, X_j]. \tag{33}$$

1.8 Mean of a function

We define the mean of a function h as

$$\mathrm{Mean}[h(X)] = \int_{\mathbb{R}} h(s) f_X(s) ds. \tag{34}$$

We also define the mean of a function when there are two random variables X and Y, i.e.

$$\text{Mean}[h(X,Y)] = \int_{\mathbb{R}\times\mathbb{R}} h(s,t) f_{X,Y}(s,t) ds dt. \tag{35}$$

In particular, we have:

$$\text{Mean}[XY] = \int_{\mathbb{R}\times\mathbb{R}} st f_{X,Y}(s,t) ds dt, \tag{36}$$

so that IN GENERAL $\text{Mean}[XY] \neq \text{Mean}[X]\,\text{Mean}[Y]$.

1.9 Normal distribution

A normal distribution is a continuous probability distribution. We say that a random variable X is normally distributed if its related probability density function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The value μ is the mean, and the value σ^2 is the variance. If $\mu = 0$ and $\sigma = 1$, we say that it is a standard normal distribution. When the mean is equal to zero, i.e. $\mu = 0$, we have that

$$\text{Mean}[X^4] = 3\sigma^4. \tag{37}$$

When the mean is not equal to zero, i.e. $\mu \neq 0$, we have that

$$\text{Mean}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4. \tag{38}$$

1.10 Brownian motion

The Brownian motion B is a random function. It means that for each event, we observe a function B_t of the time $t \geq 0$ which depends on the event. It is a generalization of normal distribution. It has many applications in economics, finance, biology, and physics. This course is focused on applications in finance. More specifically, we consider that the Brownian motion represents the price of a financial stock, such as Mitsubishi. This stock price evolves for one day. The reason why we need a random function to model is due to the fact that prices of stock change very fast, such as one second, or even one millisecond nowadays. Traditional time series models, based on a fixed increment of time, cannot be used for this problem. Figure 2 shows the function of the Brownian motion on one event.

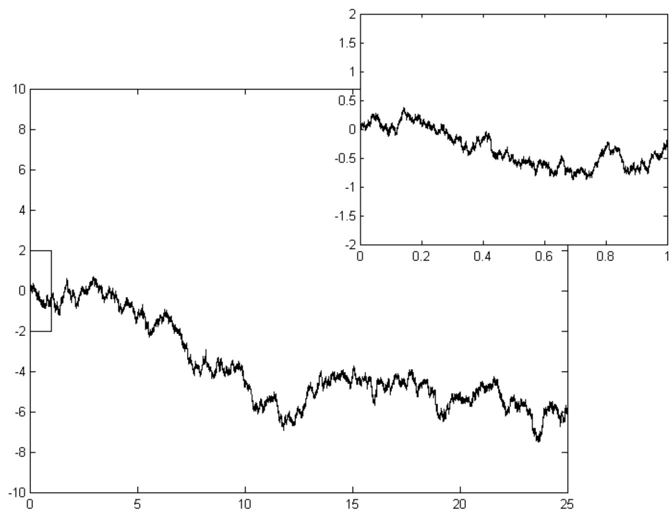

Figure 2: Function of the Brownian motion on one event

We give the mathematical properties of the Brownian motion.

First, the Brownian motion starts from the origin 0, i.e.

$$B_0 = 0. \tag{39}$$

Second, the increments of a Brownian motion have a normal distribution with mean equal to zero, and variance equal to the time increment, i.e.

$$B_{t+u} - B_t \text{ have a normal distribution with mean=0, variance=} u \text{ for any } u \geq 0. \tag{40}$$

In particular, the first and second property imply the following property.

$$\text{The Brownian motion } B_t \text{ has a normal distribution with mean equal to 0} \\ \text{and variance equal to } t \text{ for any time } t \geq 0. \tag{41}$$

Third, the Brownian motion has independent increments, i.e.

$$B_{t+u} - B_t \text{ are independent of } B_{s+v} - B_s \text{ for any } u \geq 0, v \geq 0, s + v \leq t. \tag{42}$$

1.11 Generalized Brownian motion

Although the Brownian motion looks very similar to a financial price, there are some limitations for applications. First, the mean of the Brownian motion is zero, and its instantaneous variance is one. Thus, we introduce the generalized Brownian motion with mean equal to μ and instantaneous variance equal to σ^2 as

$$X_t = \mu + \sigma B_t. \tag{43}$$

2 Exercises

2.1 Statistics

The common framework of statistics is that we observe $X_1, X_2, \cdots, X_n$ which are independent and identically distributed (with the same distribution), random variables. The number n corresponds to the number of observations. We denote the mean of X_1 as μ, i.e., $\text{Mean}[X_1] = \mu$, and the variance of X_1 as σ^2, i.e., $\text{Var}[X_1] = \sigma^2$. We note that the mean (or variance) of X_1 is equal to the mean (or variance) of X_k for any $k = 2, \ldots, n$ since they are identically distributed. We want to learn the mean μ and the variance σ^2, which are unknown. We estimate the mean μ with the average as

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Taking the average of all the observations is natural to estimate the mean μ. The average is a random variable, whereas the mean μ is a constant.

1. Calculate the mean of the average, i.e. calculate $\text{Mean}[\hat{\mu}]$.
2. Calculate the limit of the mean of the average $\text{Mean}[\hat{\mu}]$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\mu}$, i.e. calculate $\text{Var}[\hat{\mu}]$.
4. Calculate the limit of the variance of the average $\text{Var}[\hat{\mu}]$ when the number of observations n increases and interpret the result.

We estimate the variance σ^2 with the average of each variable minus the average, squared, i.e.

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2.$$

Taking the average of all the observations squared is natural to estimate the variance σ^2. Since we remove the average from each variable X_i, we obtain that the mean of $(X_i - \hat{\mu})^2$ is not equal to the variance σ^2, but a bigger number.

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate $\text{Mean}[\hat{\sigma}^2]$.

2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.

2.2 Normal distribution

We consider that the mean is zero, i.e. $\mu = 0$. We can estimate the variance σ^2 as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Taking the average of all the observations squared is natural to estimate the variance σ^2, since the mean is equal to 0.

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate $\text{Mean}[\hat{\sigma}^2]$.

2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.

3. Calculate the variance of $\hat{\sigma}^2$, i.e. calculate $\text{Var}[\hat{\sigma}^2]$.

4. Calculate the limit for the variance of the variance estimator $\text{Var}[\hat{\sigma}^2]$ when the number of observations n increases and interpret the result.

2.3 Bernoulli distribution

A Bernoulli distribution is a discrete probability distribution which takes the value 1 with probability p and the value 0 with probability $1 - p$, i.e.

$$\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0).$$

There are many applications of the Bernoulli distribution. For instance, it can be used in the head or tail coin experiment, yes or no poll questionnaires. It can be shown that

$$\mu = \text{Mean}[X] = p \quad (44)$$

$$\sigma^2 = \text{Var}[X] = p(1-p) \quad (45)$$

$$\text{Mean}[X^3] = p \quad (46)$$

$$\text{Mean}[X^4] = p \quad (47)$$

We can estimate the mean μ with the average as

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

1. Calculate the mean of the average , i.e. calculate $\text{Mean}[\hat{\mu}]$.
2. Calculate the limit of the mean of the average $\text{Mean}[\hat{\mu}]$ when the number of observations n increases and interpret the result.

By Equation (45), we can estimate the variance as

$$\hat{\sigma}^2 = \hat{\mu}(1-\hat{\mu}).$$

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate $\text{Mean}[\hat{\sigma}^2]$.
2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\sigma}^2$ when $n = 2$, i.e. calculate $\text{Var}[\hat{\sigma}^2]$.

2.4 Understanding Brownian motion

We first play with the statistical properties of the Brownian motion. The goal is to understand that the Brownian motion is both a function of time and random.

1. Calculate the mean of the Brownian motion at time t, i.e. calculate $\text{Mean}[B_t]$.
2. Calculate the limit for the mean of the Brownian motion when the time t increases and interpret the result.

3. Calculate the variance of the Brownian motion at time t, i.e. calculate $\text{Var}[B_t]$.

4. Calculate the limit for the variance of the Brownian motion when the time t increases and interpret the result.

5. Calculate the covariance of the Brownian motion at time t and time s, i.e. calculate $\text{Cov}[B_t, B_s]$ when $0 < t < s$.

6. Calculate the mean of the squared Brownian motion at time t, i.e. calculate $\text{Mean}[B_t^2]$.

7. Calculate the mean of the fourth power of the Brownian motion at time t, i.e. calculate $\text{Mean}[B_t^4]$.

2.5 Understanding the generalized Brownian motion

We now play with the statistical properties of the generalized Brownian motion.

1. Calculate the mean of the generalized Brownian motion at time t, i.e. calculate $\text{Mean}[X_t]$.

2. Calculate the limit for the mean of the generalized Brownian motion when the time t increases and interpret the result.

3. Calculate the variance of the generalized Brownian motion at time t, i.e. calculate $\text{Var}[B_t]$.

4. Calculate the limit for the variance of the generalized Brownian motion when the time t increases and interpret the result.

5. Calculate the covariance of the generalized Brownian motion at time t and time s, i.e. calculate $\text{Cov}[X_t, X_s]$ when $0 < t < s$.

6. Calculate the mean of the squared generalized Brownian motion at time t, i.e. calculate $\text{Mean}[X_t^2]$.

7. Calculate the mean of the fourth power of the generalized Brownian motion at time t, i.e. calculate $\text{Mean}[X_t^4]$.

2.6 Statistics for generalized Brownian motion

We consider an important problem in statistics for generalized Brownian motion. Depending on the point-of-view, this can also be seen as a problem for financial econometrics. It is based on financial data.

We first observe the generalized Brownian motion everyday, i.e. we observe X_i when $i = 0, 1, \cdots, n$. We want to learn the mean μ and the variance σ^2, which are unknown. We estimate the mean μ with the average as

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Taking the average of all the observations is natural to estimate the mean μ. The average is a random variable, whereas the mean μ is a constant.

1. Calculate the mean of the average, i.e. calculate $\text{Mean}[\hat{\mu}]$.
2. Calculate the limit of the mean of the average $\text{Mean}[\hat{\mu}]$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\mu}$, i.e. calculate $\text{Var}[\hat{\mu}]$.
4. Calculate the limit of the variance of the average $\text{Var}[\hat{\mu}]$ when the number of observations n increases and interpret the result.

We can estimate the variance σ^2 as

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - X_{i-1})^2.$$

Taking the average of the observation at time i minus the observation at time $i - 1$, squared, is natural to estimate the variance σ^2.

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate $\text{Mean}[\hat{\sigma}^2]$.
2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\sigma}^2$, i.e. calculate $\text{Var}[\hat{\sigma}^2]$.
4. Calculate the limit for the variance of the variance estimator $\text{Var}[\hat{\sigma}^2]$ when the number of observations n increases and interpret the result.