Notes of *Stochastic Calculus, An introduction with Application*

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1 Probability space

We start by introducing the theoretical framework of probability theory. If this is too abstract for the reader, then he/she should directly jump to the next section. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of three ingredients:

1. A sample space Ω, which is the set of all possible outcomes.
2. An event space $\mathcal{F}$, which is a set of outcomes in the sample space. In particular, we have $\mathcal{F} \subset P(\Omega)$, where P the set of all the subsets of Ω.
3. A probability function $\mathbb{P}$, which assigns each event in the event space a probability, which is a number between 0 and 1.

In general, there is no specification on Ω, so it is a "general" space, but it can happen that it corresponds to the six possible outcomes of a dice roll, etc.

2 Continuous random variable

A random variable is a function $X \colon \Omega \to \mathbb{R}$, where $\mathbb{R} := (-\infty, \infty)$ is the space of real numbers. The probability that X takes on a value in $S \subset \mathbb{R}$ is defined as

$$\mathbb{P}(X \in S) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in S\}). \tag{1}$$

For example, X can be equal to the current price of Mitsubishi asset, the current temperature in Tokyo, etc.
We say that X has a related density f_X if for any two real numbers $a \leq b$, we have:

$$\mathbb{P}(X \in [a,b]) = \int_a^b f_X(s)ds. \tag{2}$$

The only density that we will be looking at in this class is the normal distribution, which we will be defining in a later section.
The definition can be generalized in the two-dimensional case, where we consider the joint density. More specifically, if X and Y are two random variables, then their joint density takes on the form $f_{X,Y}(s,t)$.

3 Independence

Two random variables are independent if the realization of one does not affect the probability distribution of the other. For example, if X stands for the current temperature in Tokyo, and Y corresponds to the number of students enrolled in this class, it would be reasonable to assume that X and Y are independent. If we add a third random variable Z, which is the current temperature in Yokohama, then it would be reasonable to assume Y and Z are independent, but it would be unreasonable to assume independence between X and Z. Indeed, if it is cold in Tokyo, it will be most likely cold in Yokohama too, and vice versa.
Formally, two random variables X and Y are independent if their joint density and their own density satisfy

$$f_{X,Y}(s,t) = f_X(s)f_Y(t) \quad \text{for any } (s,t) \in \mathbb{R} \times \mathbb{R}. \tag{3}$$

4 Expectation

Now we define the expectation, which is a NUMBER and not a random variable, for a general function $h : \mathbb{R} \to \mathbb{R}$. Cases of interest for the reader are mean (i.e. when $h(s) = s$) and variance (i.e. when $h(s) = (s - \mathbb{E}[X])^2$), which will be specified after the general definition. The formal definition of expectation is given as

$$\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(s) f_X(s) ds. \tag{4}$$

In particular, the mean of X is specified as

$$\mathbb{E}[X] = \int_{\mathbb{R}} s f_X(s) ds, \tag{5}$$

and the variance of X as

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{\mathbb{R}} (s - \mathbb{E}[X])^2 f_X(s) ds. \tag{6}$$

We also provide the definition of expectation, when there are two random variables X and Y, and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In that case, we have:

$$\mathbb{E}[h(X, Y)] = \int_{\mathbb{R}\times\mathbb{R}} h(s, t) f_{X,Y}(s, t) ds dt. \tag{7}$$

In particular, we have:

$$\mathbb{E}[XY] = \int_{\mathbb{R}\times\mathbb{R}} st f_{X,Y}(s, t) ds dt, \tag{8}$$

so that IN GENERAL $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. We also give the definition of covariance:

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \int_{\mathbb{R}\times\mathbb{R}} (s - \mathbb{E}[X])(t - \mathbb{E}[Y]) f_{X,Y}(s, t) ds dt. \tag{9}$$

We provide now some very useful properties about expectation and variance:

$$\begin{aligned}
\mathbb{E}[X + Y] &= \mathbb{E}[X] + \mathbb{E}[Y] && (10)\\
\mathbb{E}[XY] &= \mathbb{E}[X]\mathbb{E}[Y] if X and Y are INDEPENDENT && (11)\\
\text{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 && (12)\\
\text{Cov}[X, Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] && (13)\\
\text{Var}[X + Y] &= \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y]. && (14)
\end{aligned}$$

5 Normal distribution

In probability theory, a (one-dimensional) normal distribution is a continuous probability distribution. We say that a random variable X is normally distributed if its related probability density function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The parameter μ is the mean or expectation of the distribution, σ, and the variance is σ^2. If $\mu = 0$ and $\sigma = 1$, we say that it is a standard normal distribution.