Final take-home exam of *ADVANCED STUDY OF ECONOMETRICS/STOCHASTIC CALCULUS, AN INTRODUCTION WITH APPLICATION/STOCHASTIC CALCULUS, AN INTRODUCTION WITH APPLICATION (GPP)*

Yoann Potiron
potiron@keio.jp
Keio University

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We consider an important problem in statistics for generalized Brownian motion. Depending on the point-of-view, this can also be seen as a problem for financial econometrics. It is based on financial data.

1 Statistics based on everyday observation of the generalized Brownian motion

We first observe the generalized Brownian motion everyday, i.e. we observe X_i when $i = 0, 1, \cdots, n$. We want to learn the mean μ and the variance σ^2, which are unknown.

1.1 Estimation of the mean

We estimate the mean μ with the average as

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Taking the average of all the observations is natural to estimate the mean μ. The average is a random variable, whereas the mean μ is a constant.

1. Calculate the mean of the average, i.e. calculate Mean$[\hat{\mu}]$.
2. Calculate the limit of the mean of the average Mean$[\hat{\mu}]$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\mu}$, i.e. calculate Var$[\hat{\mu}]$.
4. Calculate the limit of the variance of the average Var$[\hat{\mu}]$ when the number of observations n increases and interpret the result.

1.2 Estimation of the variance

We can estimate the variance σ^2 as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{i-1})^2.$$

Taking the average of the observation at time i minus the observation at time $i-1$, squared, is natural to estimate the variance σ^2.

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate Mean$[\hat{\sigma}^2]$.
2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.
3. Calculate the variance of $\hat{\sigma}^2$, i.e. calculate Var$[\hat{\sigma}^2]$.
4. Calculate the limit for the variance of the variance estimator Var$[\hat{\sigma}^2]$ when the number of observations n increases and interpret the result.

2 Statistics based on observations during one day of the generalized Brownian motion

We observe now the generalized Brownian motion during one day. This is also called observations based on high frequency. By observations based on high frequency, we mean all the observations recorded from one day of trading. This day of trading is represented as the interval $[0, 1]$. We observe the generalized Brownian motion X_t at times t_i for $i = 0, \ldots, n$. Then, we have $n + 1$ observations of the generalized Brownian motion. These observations are at equal distance from each other where $\Delta = \frac{1}{n}$, i.e. $t_k = k\Delta$ for $k = 0, \cdots, n$. In other words, we observe the generalized Brownian motion at times $t_0 = 0, t_1 = \frac{1}{n}, t_2 = \frac{2}{n}, \cdots, t_n = 1$. The bigger n is, the more observations we get during the day.

We want to learn the variance σ^2. We can estimate the variance σ^2 as

$$\hat{\sigma}^2 = \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2.$$

Taking the sum of the observation at time t_i minus the observation at time t_{i-1}, squared, is natural to estimate the variance σ^2.

1. Calculate the mean of the variance estimator $\hat{\sigma}^2$, i.e. calculate $\text{Mean}[\hat{\sigma}^2]$.

2. Calculate the limit for the mean of the variance estimator $\hat{\sigma}^2$ when the number of observations n increases and interpret the result.

3. Calculate the variance of $\hat{\sigma}^2$, i.e. calculate $\text{Var}[\hat{\sigma}^2]$.

4. Calculate the limit for the variance of the variance estimator $\text{Var}[\hat{\sigma}^2]$ when the number of observations n increases and interpret the result.