First passage time and inverse problem for continuous local martingales

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Abstract: This paper derives an explicit formula for the probability that a continuous local martingale crosses a one or two-sided random constant boundary for a finite time interval. The boundary crossing probability of a continuous local martingale to a constant boundary is equal to the boundary crossing probability of a standard Wiener process, which is time-changed by the martingale quadratic variation, to a constant boundary. This paper also derives an explicit solution to the inverse first passage time problem of quadratic variation. These results are obtained by an application of the Dambis, Dubins-Schwarz theorem. The main elementary idea of the proof is the scale invariant property of the time-changed Wiener process and thus the scale invariant property of the first passage time. This is due to the constancy of the boundary.

Keywords and phrases: Mathematical statistics, applied probability, first passage time problem, inverse first passage time problem, boundary crossing probabilities.

1. Introduction

This paper first concerns the boundary crossing probabilities of the first passage time (FPT), i.e., the probability that a stochastic process crosses a boundary. This paper also concerns an inverse first passage time (IFPT) problem. The IFPT problem determines the increasing function such that the FPT of a standard Wiener process, which is time changed by this increasing function, to the boundary has a given distribution. As far as the author knows, this IFPT is completely new to the literature. This problem differs from the Shiryaev IFPT problem, which determines the boundary function such that the FPT of a standard Wiener process to this boundary has a given distribution.

The application of the FPT in statistics can be traced back to the Kolmogorov-Smirnov statistic. The primary application of the FPT can be found in sequential analysis. At first, the focus was on the FPT of a random walk. Due to the complexity of the problem, the literature often relies on the FPT of a Wiener process (see Gut (1974), Woodroofe (1976), Woodroofe (1977)), Lai and Siegmund (1977)), Lai and Siegmund (1979)) and Siegmund (1986)). In survival analysis, Matthews, Farewell and Pyke (1985) show that tests for constant hazard involve the FPT of an Ornstein-Uhlenbeck process. Butler and Huzurbazar (1997)) consider a Bayesian approach for the FPT of a semi-Markovian process. Eaton and Whitmore (1977) study the application of the FPT for hospital stay. Aalen and Gjessing (2001)) consider the FPT of a Markovian process. Detailed reviews on the FPT can be found in Lee and Whitmore (2006) and Lawless (2011) (see Section 11.5, pp. 518-523). Finally, Roberts and Shortland (1997) and Borovkov and Novikov (2002) provide an application of the FPT for the pricing of barrier options in mathematical finance.

Although the IFPT problem is new, it has useful applications in financial econometrics. More specifically, the FPT of a continuous local martingale have applications when estimating the quadratic variation of a continuous local martingale based on endogenous observations. In these models, endogenous observations are often generated by the FPT of a local martingale to a boundary process. Fukasawa (2010) considers the FPT to a symmetric two-sided boundary. Robert and Rosenbaum (2011) and Robert and Rosenbaum (2012) (see also Section 4.4 in Potiron and Mykland (2020) for its extension) introduce the model with uncertainty zones in which the two-sided boundary is dynamic. Fukasawa and Rosenbaum (2012) consider the FPT to a two-sided boundary, which is non-symmetric. Abbring (2012) studies the mixed FPT of a Levy process. Renault, Van der Heijden and Werker (2014) consider the mixed FPT of the sum of a Wiener process and a positive linear drift. Finally, Potiron and Mykland (2017) estimate the quadratic covariation between two local martingales.

In these examples, two natural questions remain. First, is there a process that generates a given distribution? Second, what is the quadratic variation of this process? With the use of the IFPT problem, we can prove the existence and determine the quadratic variation of the stochastic process. Kikuchi, Li and Potiron (2026) consider nonparametric estimation of the explicit solution in the IFPT problem. In their empirical application to financial data, they find that the quadratic variation is not linear. However, most of the literature focuses on a standard Wiener process, which has by definition a linear quadratic variation. This is the reason why we consider a continuous local martingale in this paper.

Explicit formulae of the boundary crossing probabilities mostly exist in the case of a Wiener process. Doob (1949) (Equations (4.2)-(4.3), pp. 397-398) obtains explicit formulae, based on elementary geometrical and analytical arguments. Malmquist (1954) (Theorem 1, p. 526) gives an explicit formula conditioned on the starting and final values of the Wiener process for a finite final time. Anderson (1960) (Theorem 4.2, pp. 178-179) derives an explicit formula conditioned on the final value of the Wiener process in the two-sided boundary case with linear drift. Then, he integrates it with respect to the final value of the Wiener process to get an explicit formula (Theorem 4.3, p. 180). For square root boundaries, Breiman (1967) rewrites the problem as the FPT of an Ornstein-Uhlenbeck process to a constant boundary. With the same technique, Daniels (1969) derives an explicit formula. Nobile, Ricciardi and Sacerdote (1985) investigate the asymptotic behaviour of the FPT by an Ornstein–Uhlenbeck process to a large constant boundary. Kou and Wang (2003) derives, in the form of Laplace transform, the boundary crossing probabilities of a jump diffusion process with linear drift to a constant boundary. Alili and Kyprianou (2005), Doney and Kyprianou (2006) and Kyprianou, Pardo and Rivero (2010) study a link between the FPT, last passage time, and overshoot above or below a fixed level of a Levy process. Borovkov and Novikov (2008) find an explicit formula for the

Laplace transform of the FPT of a Levy-driven Ornstein–Uhlenbeck process to a two-sided constant boundary. Potiron (2025) derives non-explicit formulae of the FPT by a Wiener process, which has a stochastic drift and random variance, to a stochastic boundary. For the inverse IFPT problem, there is as far as the author knows no related paper on it, since this is a new problem.

In this paper, we first derive an explicit formula for the one-sided and two-sided boundary crossing probability of a continuous local martingale to a constant boundary. We derive the results in two cases: (i) a nonrandom case when the boundary is nonrandom constant and the quadratic variation of the continuous local martingale is a nonrandom time-dependent function and (ii) a random case when the boundary is random constant and the quadratic variation of the continuous local martingale is a stochastic process.

We consider a continuous local martingale Z and its quadratic variation defined as $\langle Z \rangle$. We also consider two boundaries g and h. We focus on the one-sided and two-sided boundary crossing probabilities defined as

$$P_g^Z(t) = \mathbb{P}\big(\sup_{0 \leq s \leq t} Z_s \geq g\big), \tag{1}$$

$$P_{g,h}^Z(t) = \mathbb{P}\big(\sup_{0 \leq s \leq t} Z_s \geq g \text{ or } \inf_{0 \leq s \leq t} Z_s \leq h\big). \tag{2}$$

These boundary crossing probabilities correspond to the probability that the process Z crosses one of both boundaries between the starting time 0 and the final time t. To reexpress the IFPT problem, we can first reexpress Z as a standard Wiener process, which is time-changed by the quadratic variation $\langle Z \rangle$. Then, we can focus on the following IFPT problem. We want to determine the quadratic variation of Z, namely $\langle Z \rangle$, such that the FPT of Z to the boundary has a given cdf F.

We introduce a standard Wiener process W. We define the boundary crossing probabilities $P_g^W(t)$ and $P_{g,h}^W(t)$ as a specification of Equations (1) and (2) in the Wiener process case. We consider first the one-sided nonrandom case. We obtain that the boundary crossing probability of a continuous local martingale to a constant boundary is equal to the boundary crossing probability of a standard Wiener process, which is time changed by the martingale quadratic variation, to a constant boundary. More specifically, we obtain that (see Theorem 1)

$$P_g^Z(t) = P_g^W\big(\langle Z \rangle_t\big).$$

This explicit formula is obtained by an application of the Dambis, Dubins-Schwarz theorem. The main elementary idea of the proof is the scale invariant property of the time-changed Wiener process and thus the scale invariant property of the FPT. This is due to the constancy of the boundary. In the two-sided nonrandom case, we obtain that (see Theorem 2)

$$P_{g,h}^Z(t) = P_{g,h}^W\big(\langle Z \rangle_t\big).$$

To apply the Dambis, Dubins-Schwarz theorem in the one-sided random case, the main elementary idea is to rewrite the FPT to a random boundary as an equivalent FPT to a nonrandom boundary. This is obtained by considering the new stochastic process as $Y = \frac{Z}{g}$ and the new boundary as 1. We also define the set of functions which are nonrandom and nondecreasing from $\mathbb{R}^+$ to $\mathbb{R}^+$ as $\mathcal{P}$ and the cumulative distribution function (cdf) of $\langle Y \rangle$ as $F_{\langle Y \rangle}$. We obtain that (see Theorem 3)

$$P_1^Y(t) \quad = \quad \int_{\mathcal{P}} P_1^W(y_t) dF_{\langle Y \rangle}(y).$$

This is obtained by regular conditional probability, and using the explicit formula obtained in the nonrandom case.

To apply the Dambis, Dubins-Schwarz theorem in the two-sided random case, we cannot rewrite the FPT to a random two-sided boundary as an equivalent FPT to a nonrandom two-sided boundary since there are two boundaries. However, we are able to adapt the arguments with a two-sided boundary. We define the triplet of two boundaries and quadratic variation as $u = (g, h, \langle Z \rangle)$, and its cdf as F_u. We also define the product of the boundaries and functions which are nonrandom and nondecreasing from $\mathbb{R}^+$ to $\mathbb{R}^+$ as $\mathcal{S} = \mathcal{J} \times \mathcal{P}$. If we assume that the stochastic process Z is independent from the two-sided boundary (g, h), we obtain that (see Theorem 4)

$$P_{g,h}^Z(t) \quad = \quad \int_{\mathcal{S}} P_{g_0,h_0}^W(z_t) dF_u(g_0, h_0, z). \tag{3}$$

In this paper, we then derive an explicit solution for the IFPT problem. We consider the one-sided and two-sided boundary with nonrandom case and random case. We also consider the explicit solution in the case when the quadratic variation is absolutely continuous and in the case when the quadratic variation is not absolutely continuous. The proofs are based on the use of the explicit formula of boundary crossing probability (1) and (2), and elementary topological arguments.

We first consider the one-sided and nonrandom case when the quadratic variation $\langle Z \rangle$ is absolutely continuous of the form $\langle Z^f \rangle_t = \int_0^t \sigma_{s,f}^2 ds$. Then, we can also define the pdf of F as f and focus on variance functions $\sigma_{t,f}^2$. We define the error function and its inverse as $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$ and erfinv. We also introduce the notation $h(t) = \text{erfinv}(1 - F(t))$. We obtain that the explicit solution of the variance function is equal to (see Theorem 5)

$$\sigma_{t,f}^2 = \frac{f(t)}{\frac{2}{g^2\sqrt{\pi}} h(t)^3 e^{-h(t)^2}} \mathbf{1}_{\{0<F(t)<1\}}.$$

In the case when the quadratic variation is not absolutely continuous, we obtain that the explicit solution is equal to (see Theorem 6)

$$\langle Z^F \rangle_t \quad = \quad \frac{g^2}{2h(t)^2} \mathbf{1}_{\{0<F(t)<1\}}.$$

We consider now the two-sided and nonrandom case when the quadratic variation is absolutely continuous. We obtain that the explicit solution is equal to (see Theorem 7)

$$\sigma^2_{t,f} = \frac{f(t)}{f^W_{g,h}((P^W_{g,h})^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}}.$$

When the quadratic variation is not absolutely continuous, we obtain that the explicit solution is equal to (see Theorem 8)

$$\langle Z^F\rangle_t = (P^W_{g,h})^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}}.$$

We consider now the one-sided and random case, in which we define F as the random cdf. When the random quadratic variation is absolutely continuous of the form $\langle Y^f\rangle_t(\omega) = \int_0^t \sigma^2_{s,f}(\omega)ds$, we can define its random pdf as f. We introduce the notation $h(t,\omega) = \text{erfinv}(1 - F(t,\omega))$. We obtain that the explicit solution is equal to (see Theorem 9)

$$\sigma^2_{t,f}(\omega) = \frac{f(t,\omega)}{\frac{2}{g^2\sqrt{\pi}}h(t,\omega)^3 e^{-h(t,\omega)^2}}\mathbf{1}_{\{0<F(t,\omega)<1\}}.$$

When the random quadratic variation is not absolutely continuous, we obtain that the explicit solution is equal to (see Theorem 10)

$$\langle Y^F\rangle_t(\omega) = \frac{1}{2h(t,\omega)^2}\mathbf{1}_{\{0<F(t,\omega)<1\}}.$$

We consider now the two-sided random case. When the random quadratic variation is absolutely continuous, we obtain that the explicit solution is equal to (see Theorem 11)

$$\sigma^2_{t,f}(\omega) = \frac{f(t,\omega)}{f^W_{g,h}((P^W_{g,h})^{-1}(F(t,\omega)))}\mathbf{1}_{\{0<F(t,\omega)<1\}}.$$

When the random quadratic variation is not absolutely continuous, we obtain that the explicit solution is equal to (see Theorem 12)

$$\langle Z^F\rangle_t(\omega) = (P^W_{g,h})^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}}.$$

The following of this paper is structured as follows. We derive an explicit formula for the boundary crossing probability in Section 2. We obtain an explicit solution for the IFPT problem in Section 3. The proofs of the explicit formula are given in Appendix A. The proofs of the explicit solution for the IFPT problem can be found in Appendix B.

2. Explicit formula of boundary crossing probability

In this section, we derive an explicit formula for the one-sided and two-sided boundary crossing probability (1)-(2) of a continuous local martingale in the nonrandom case and random case.

2.1. One-sided nonrandom case

In this part, we consider the case when the one-sided boundary is nonrandom and constant, and the quadratic variation of the continuous local martingale is a nonrandom time-dependent function.

We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \Sigma, \mathcal{F})$, where Σ is a σ-field and $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is a filtration. We define the set without 0 as $\mathcal{S}_*^+$ for any set $\mathcal{S}$. Then, we define the set of nonrandom constant functions from the space $\mathbb{R}^+$ to the space $\mathbb{R}_*$ as $\mathbb{R}_*$. By a constant function $g \in \mathbb{R}_*$, we mean that $g(t) = g(u)$ for any time $t \in \mathbb{R}^+$ and any time $s \in \mathbb{R}^+$. We first give the definition of the set of boundary functions.

Definition 1. We define the set of boundary functions which are nonrandom, constant and one-sided as $\mathbb{R}_*$.

We introduce an $\mathcal{F}$-adapted continuous stochastic process Z with starting time 0. With these assumptions, we can even consider a stochastic process which does not start from the origin 0 and a boundary which is nonpositive, if they satisfy $Z_0 < g$ with a nonrandom Z_0. Then, we can reexpress the new process, with starting value 0, as $Z_t - Z_0$ for any time $t \geq 0$. We can also reexpress the new positive boundary as $g - Z_0$. We now give the definition of the FPT.

Definition 2. We define the FPT of the stochastic process Z to a boundary $g \in \mathbb{R}_*$ as

$$\mathrm{T}_g^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g\} \text{ for } \omega \in \Omega. \tag{4}$$

We have that Z is a continuous and $\mathcal{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } (t, Z_t) \in G\}$, where $G = \{(t,u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u \geq g\}$ is a closed subset of $\mathbb{R}^2$. Thus, the FPT T_g^Z is an $\mathcal{F}$-stopping time by Theorem I.1.27 (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability P_g^Z as the cdf of T_g^Z, i.e.,

$$P_g^Z(t) = \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for } t \geq 0. \tag{5}$$

If the cdf is absolutely continuous, we can also define its pdf $f_g^Z : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f_g^Z(t) \quad = \quad \frac{dP_g^Z(t)}{dt} \text{ for } t \geq 0 \text{ a.e..} \tag{6}$$

We introduce an $\mathcal{F}$-adapted standard Wiener process W. We define the standard normal cdf as

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} exp\big(-\frac{u^2}{2}\big) du \text{ for } t \in \mathbb{R}. \tag{7}$$

We first consider the case when the stochastic process is a standard Wiener process, i.e., when $Z_t = W_t$ for any time $t \in \mathbb{R}^+$. The next lemma gives an explicit formula of P_g^W and f_g^W, namely a Levy distribution. These are known results by integrating the explicit formula conditioned on the final value of the Wiener process (see Malmquist (1954), p. 526) with respect to the Wiener process final value (see Wang and Pötzelberger (1997), Equations (3), p. 55).

Lemma 1. *We obtain a Levy distribution with $P_g^W(0) = 0$, $f_g^W(0) = 0$. We also obtain*

$$P_g^W(t) = 1 - \Phi\left(\frac{g}{\sqrt{t}}\right) + \Phi\left(\frac{-g}{\sqrt{t}}\right) \textit{ for } t > 0, \tag{8}$$

$$f_g^W(t) = \frac{g}{\sqrt{2\pi t^3}} e^{\frac{-g^2}{2t}} \textit{ for } t > 0. \tag{9}$$

The explicit formula in the one-sided nonrandom case, i.e. Theorem 1, states that the boundary crossing probability of a continuous local martingale to a constant boundary is equal to the boundary crossing probability of a standard Wiener process, which is time-changed by the martingale quadratic variation, to a constant boundary. We get a time-changed Levy cdf. This is obtained by an application of the Dambis, Dubins-Schwarz theorem for continuous local martingale (see, Revuz and Yor (2013), Th. V.1.6). Accordingly, we provide the assumption on the continuous local martingale which is required to apply the Dambis, Dubins-Schwarz theorem.

Assumption 1. We assume that Z is a continuous $\mathcal{F}$-adapted local martingale with nonrandom quadratic variation $\langle Z \rangle$ and such that $Z_0 = 0$ a.s. and $\langle Z \rangle_\infty = \infty$ a.s..

For a function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $a \mapsto h(a)$ and a set $A \subset \mathbb{R}^+$, we define the restriction of h to A as $h \upharpoonright_A$ such that $h \upharpoonright_A : A \to \mathbb{R}^+$ with $a \mapsto h(a)$. For a measurable set $A \subset \mathbb{R}^+$ and $p \in \mathbb{R}$ with $p \geq 1$, we define the set of p-integrable and nonrandom functions as

$$L_p(A) = \big\{ h : A \to \mathbb{R}^+ \text{ measurable s.t. } \int_A |h(x)|^p \, \mathrm{d}x < +\infty \big\}.$$

For a measurable $A \subset \mathbb{R}^+$ and $p \in \mathbb{R}$ with $p \geq 1$, we define the set of locally p-integrable and nonrandom functions as

$$\begin{aligned} L_{p,\text{loc}}(A) = \big\{ & h : A \to \mathbb{R}^+ \text{ measurable s.t.} \\ & h \upharpoonright_K \in L_p(K) \; \forall \, K \subset A, \, K \text{ compact} \big\}. \end{aligned}$$

In the following example, we show that a continuous $\mathcal{F}$-Itô process satisfies Assumption 1.

Example 1. We can consider a continuous $\mathcal{F}$-Itô process with no trend, i.e.

$$Z_t = \int_0^t \sigma_s dW_s \text{ for } t \geq 0. \tag{10}$$

Here, the standard deviation $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonrandom function. If we assume that $\sigma \in L_{2,\text{loc}}(\mathbb{R}^+)$, then Z is an $\mathcal{F}$-local martingale with nonrandom quadratic variation $\langle Z \rangle_t = \int_0^t \sigma_u^2 du$ by Theorem I.4.40 (p. 48) in Jacod and Shiryaev (2003). If we further assume that the variance integral satisfies a.s. $\int_0^t \sigma_u^2 du \to \infty$ as $t \to \infty$, we have that Z satisfies Assumption 1.

We state Theorem 1 in what follows.

Theorem 1. *Under Assumption 1, we have that*

$$P_g^Z(t) \quad = \quad P_g^W\big(\langle Z\rangle_t\big) \text{ for } t \geq 0. \tag{11}$$

As a corollary, we obtain the pdf from the FPT in the one-sided nonrandom case if we assume that the quadratic variation is absolutely continuous. Then, there exists a derivative, which we define as $\langle Z\rangle_t'$ for $t \geq 0$ a.e.

Corollary 1. *Under Assumption 1 and if we assume that the quadratic variation $\langle Z\rangle$ is absolutely continuous on $\mathbb{R}^+$, we have that*

$$f_g^Z(t) \quad = \quad \langle Z\rangle_t' f_g^W\big(\langle Z\rangle_t\big) \text{ for } t \geq 0 \text{ a.e.} \tag{12}$$

2.2. Two-sided nonrandom case

In this part, we consider the case when the two-sided boundary is nonrandom constant and the quadratic variation of the continuous local martingale is a nonrandom time-dependent function.

We first give the definition of the set of boundary functions.

Definition 3. We define the set of boundary functions which are nonrandom, constant and two-sided as $(\mathbb{R}_*^+, \mathbb{R}_*^-)$.

We consider an $\mathcal{F}$-adapted continuous stochastic process Z, started at the origin point 0. With these assumptions, we can even consider a process which does not start from 0 and boundaries which are nonpositive and nonnegative if they satisfy $h < Z_0 < g$ with a nonrandom Z_0. Then, we can reexpress the new process started at 0 as $Z_t - Z_0$ for any time $t \geq 0$. We can also reexpress the new positive boundary as $g - Z_0$ and the new negative boundary as $h - Z_0$. We now give the definition of the FPT.

Definition 4. We define the FPT of the process Z to a boundary $(g, h) \in (\mathbb{R}_*^+, \mathbb{R}_*^-)$ as

$$\mathrm{T}_{g,h}^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g \text{ or } Z_t \leq h\} \text{ for } \omega \in \Omega. \tag{13}$$

We have that Z is a continuous and $\mathcal{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g \text{ or } Z_t \leq h\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } (t, Z_t) \in G\}$, where $G = \{(t, u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u \geq g \text{ or } u \leq h\}$ is a closed subset of $\mathbb{R}^2$. Thus, the FPT T_g^Z is an $\mathcal{F}$-stopping time by Theorem I.1.27 (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability $P_{g,h}^Z$ as the cdf of $\mathrm{T}_{g,h}^Z$, i.e.

$$P_{g,h}^Z(t) = \mathbb{P}(\mathrm{T}_{g,h}^Z \leq t) \text{ for } t \geq 0. \tag{14}$$

If the cdf is absolutely continuous, we can also define its pdf $f_{g,h}^Z : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f_{g,h}^Z(t) \quad = \quad \frac{dP_{g,h}^Z(t)}{dt} \text{ for } t \geq 0 \text{ a.e.} \tag{15}$$

Moreover, we define $\mathrm{ss}_t(v, w)$ for any $0 < v < w$ as

$$\mathrm{ss}_t(v,w) = \sum_{k=-\infty}^{\infty} \frac{w - v + 2kw}{\sqrt{2\pi} t^{3/2}} e^{-(w-v+2kw)^2/2t}. \tag{16}$$

We first consider the case when the stochastic process is a standard Wiener process, i.e. when $Z_t = W_t$ for any time $t \in \mathbb{R}^+$. The next lemma gives an explicit formula of $P_{g,h}^W$ and $f_{g,h}^W$ which are respectively known results from Theorem 4.3 (p. 180) in Anderson (1960).

Lemma 2. *We obtain that $P_{g,h}^W(0) = 0$, $f_{g,h}^W(0) = 0$. We also obtain that*

$$\begin{aligned} P_{g,h}^W(t) &= \sum_{k=-\infty}^{\infty} \Big(4 - \\ &\quad 2\Phi\left(\frac{-h + 2k(g-h)}{\sqrt{t}}\right) - 2\Phi\left(\frac{g + 2k(g-h)}{\sqrt{t}}\right)\Big) \text{ for } t > 0, \end{aligned} \tag{17}$$

$$f_{g,h}^W(t) = \mathrm{ss}_t\left(g, g-h\right) + \mathrm{ss}_t\left(-h, g-h\right) \text{ for } t > 0. \tag{18}$$

The explicit formula in the two-sided nonrandom case, namely Theorem 2, is obtained by an application of the Dambis, Dubins-Schwarz theorem. Indeed, the arguments used in the one-sided boundary case extend directly to this two-sided boundary case.

Theorem 2. *Under Assumption 1, we have that*

$$P_{g,h}^Z(t) = P_{g,h}^W\big(\langle Z\rangle_t\big) \text{ for } t \geq 0. \tag{19}$$

As a corollary, we obtain the pdf from the FPT of a continuous local martingale to a constant boundary if we assume that the quadratic variation is absolutely continuous.

Corollary 2. *Under Assumption 1 and if we assume that the quadratic variation $\langle Z\rangle$ is absolutely continuous on $\mathbb{R}^+$, we have that*

$$f_{g,h}^Z(t) = \langle Z\rangle_t' f_{g,h}^W\big(\langle Z\rangle_t\big) \text{ for } t \geq 0 \text{ a.e.}. \tag{20}$$

2.3. One-sided random case

In this part, we consider the case when the one-sided boundary is random constant and the quadratic variation of the continuous local martingale is a stochastic process.

We first give the definition of the set of boundary functions which are random variables.

Definition 5. We define the set of boundary functions which are random, constant, one-sided and $\mathcal{F}$-adapted as $g(\omega) \in \mathbb{R}_*$ for any $\omega \in \Omega$.

We consider an $\mathcal{F}$-adapted continuous stochastic process Z started from the origin 0. With these assumptions, we can even consider a stochastic process which does not start from 0 and a boundary which is nonpositive, if they satisfy $\mathbb{P}(Z_0 < g) = 1$. Then, we can reexpress the new stochastic process, started from the origin 0, as $Z_t - Z_0$ for any time $t \geq 0$ a.s. We can also reexpress the new positive boundary as $g - Z_0$ a.s. We now give the definition of the FPT.

Definition 6. We define the FPT of the stochastic process Z to a random boundary g as

$$\mathrm{T}_g^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g\}. \tag{21}$$

We have that Z/g is a continuous and $\mathcal{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } (t, Z_t/g) \in G\}$, where $G = \{(t,u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u \geq 1\}$ is a closed subset of $\mathbb{R}^2$. Thus, the FPT T_g^Z is an $\mathcal{F}$-stopping time by Theorem I.1.27 (p. 7) in Jacod and Shiryaev (2003). We can rewrite the boundary crossing probability P_g^Z as the cdf of T_g^Z, i.e.,

$$P_g^Z(t) = \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for } t \geq 0. \tag{22}$$

If the cdf is absolutely continuous, we can also define its pdf $f_g^Z : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f_g^Z(t) \quad = \quad \frac{dP_g^Z(t)}{dt} \text{ for } t \geq 0 \text{ a.e.}. \tag{23}$$

The explicit formula in the one-sided random case, namely Theorem 3, is obtained by an application of the Dambis, Dubins-Schwarz theorem. To apply the theorem in the random case, the main elementary idea is to rewrite the FPT to a random boundary as an equivalent FPT to a nonrandom boundary. This is obtained by dividing both the stochastic process and the boundary by the boundary value. More specifically, we define the new process as $Y = \frac{Z}{g}$ and the new boundary as 1. Then, we observe that the FPT (21) may be rewritten as

$$\mathrm{T}_g^Z = \mathrm{T}_1^Y. \tag{24}$$

In what follows, we give the assumption on the stochastic process Y.

Assumption 2. We assume that Y is a continuous $\mathcal{F}$-local martingale with random quadratic variation $\langle Y \rangle$ and such that $Y_0 = 0$ a.s. and $\langle Y \rangle_\infty = \infty$ a.s.

We introduce a stochastic process $h : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ defined as $a \mapsto h(a)$. For any $A \subset \mathbb{R}^+ \times \Omega$, we define the restriction of h to A as $h \restriction_A$ such that $h \restriction_A : A \to \mathbb{R}^+$ and $h \restriction_A (a) = h(a)$ for any $a \in A$. For $A \subset \mathbb{R}^+ \times \Omega$ measurable and $p \in \mathbb{R}$ satisfying $p \geq 1$, we define the set of stochastic processes which are p-integrable as

$$L_p(A) \quad = \quad \left\{h : A \to \mathbb{R}^+ \text{ measurable s.t. } \int_A |h(x)|^p \, \mathrm{d}x < +\infty\right\}.$$

For $A \subset \mathbb{R}^+ \times \Omega$ measurable and $p \in \mathbb{R}$ satisfying $p \geq 1$, we define the set of stochastic processes which are locally p-integrable as

$$\begin{aligned} L_{p,\mathrm{loc}}(A) &= \big\{h : A \to \mathbb{R}^+ \text{ measurable s.t.} \\ &\quad h \restriction_K \in L_p(K) \text{ for any } K \subset A,\, K \text{ compact}\big\}. \end{aligned}$$

In the following example, we rely on a continuous Itô process.

Example 2. We can consider a continuous $\mathcal{F}$-Itô process with no trend, i.e.

$$Y_t = \int_0^t \sigma_s dW_s \text{ for } t \geq 0. \tag{25}$$

Here, $\sigma : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ is an $\mathcal{F}$-predictable stochastic process such that the integral defined in Equation (25) is well-defined. If we assume that $\sigma \in L_{2,\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$, then Y is a local martingale with random quadratic variation $\langle Y \rangle_t = \int_0^t \sigma_u^2 du$ by Theorem I.4.40 (p. 48) in Jacod and Shiryaev (2003). If we further assume that the variance integral satisfies a.s. $\int_0^t \sigma_u^2 du \to \infty$ as $t \to \infty$, we have that Y satisfies Assumption 2.

We define the set of functions which are nonrandom and nondecreasing from $\mathbb{R}^+$ to $\mathbb{R}^+$ as $\mathcal{P}$. When seen as a function of ω, the arrival space of $\langle Y \rangle$ is $\mathcal{P}$. We define the distribution of $\langle Y \rangle$ as $F_{\langle Y \rangle}$. We get P_1^Y in the next theorem by regular conditional probability, and using the explicit formula obtained in the nonrandom case.

In what follows, we state Theorem 3.

Theorem 3. *Under Assumption 2, we have that*

$$P_1^Y(t) \quad = \quad \int_{\mathcal{P}} P_1^W(y_t) dF_{\langle Y \rangle}(y) \text{ for } t \geq 0. \tag{26}$$

As a corollary, we obtain the pdf from the FPT of a continuous local martingale to a constant boundary if we assume that the quadratic variation $\langle Y \rangle$ is absolutely continuous a.s.. Then, there exists derivatives to $\langle Y \rangle = y$ which we define as y'_t for $t \geq 0$ a.e. and a.s.

Corollary 3. *Under Assumption 2 and if we assume that the quadratic variation $\langle Y \rangle$ is absolutely continuous on $\mathbb{R}^+$ a.s., we have that*

$$f_1^Y(t) \quad = \quad \int_{\mathcal{P}} y'_t f_1^W(y_t) dF_{\langle Y \rangle}(y) \text{ for } t \geq 0 \text{ a.e.} \tag{27}$$

2.4. Two-sided random case

In this part, we consider the case when the two-sided boundary is random constant and the quadratic variation of the continuous local martingale is a stochastic process.

We first give the definition of the set of boundary functions which are random variables.

Definition 7. We define the set of boundary functions which are random, constant, two-sided and $\mathcal{F}$-adapted as $g(\omega) \in \mathbb{R}_*^+$ and $h(\omega) \in \mathbb{R}_*^-$ for any $\omega \in \Omega$.

We consider an $\mathcal{F}$-adapted continuous stochastic process Z, started from the origin 0. With these assumptions, we can even consider a stochastic process which does not start from 0 and boundaries which are nonpositive and nonnegative, if they satisfy $\mathbb{P}(h < Z_0 < g) = 1$. Then, we can reexpress the new process, started from 0, as $Z_t - Z_0$ for any time $t \geq 0$ a.s. We can also reexpress the new positive boundary as $g - Z_0$ a.s. and the new negative boundary as $h - Z_0$ a.s. We give the definition of the FPT in what follows.

Definition 8. We define the FPT of the stochastic process Z to a boundary (g, h) as

$$\mathrm{T}_{g,h}^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g \text{ or } Z_t \leq h\}. \tag{28}$$

We can rewrite $\mathrm{T}_{g,h}^Z$ as the infimum of two $\mathcal{F}$-stopping times, namely $\mathrm{T}_{g,h}^Z = \inf(\mathrm{T}_g^Z, \mathrm{T}_{-h}^{-Z})$. Thus, it is an $\mathcal{F}$-stopping time. We can rewrite the boundary crossing probability $P_{g,h}^Z$ as the cdf of $\mathrm{T}_{g,h}^Z$, i.e.

$$P_{g,h}^Z(t) = \mathbb{P}(\mathrm{T}_{g,h}^Z \leq t) \text{ for } t \geq 0. \tag{29}$$

If the cdf is absolutely continuous, we can also define its pdf $f_{g,h}^Z : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f_{g,h}^Z(t) = \frac{dP_{g,h}^Z(t)}{dt}. \tag{30}$$

The explicit formula in the one-sided random case, namely Theorem 4, is obtained by an application of the Dambis, Dubins-Schwarz theorem. To apply the theorem in the two-bounded random case, we cannot rewrite the FPT to a random two-sided boundary as an equivalent FPT to a nonrandom two-sided boundary since there are two boundaries. However, we are able to adapt the arguments with a two-sided boundary. We now give the assumption on the stochastic process Z.

Assumption 3. We assume that Z is a continuous $\mathcal{F}$-local martingale with random quadratic variation $\langle Z \rangle$ and such that $Z_0 = 0$ a.s. and $\langle Z \rangle_\infty = \infty$ a.s..

We define the triplet of two boundaries and quadratic variation as $u = (g, h, \langle Z \rangle)$, and its cdf as F_u. We also define the product of the boundaries and functions which are nonrandom and nondecreasing from $\mathbb{R}^+$ to $\mathbb{R}^+$ as $\mathcal{S} = \mathcal{J} \times \mathcal{P}$. We get $P_{g,h}^Z$ in the next theorem by regular conditional probability, and using the explicit formula obtained in the nonrandom case.

We state Theorem 4 in what follows.

Theorem 4. *Under Assumption 3, we have that*

$$P_{g,h}^Z(t) = \int_{\mathcal{S}} P_{g_0,h_0}^W(z_t) dF_u(g_0, h_0, z) \text{ for } t \geq 0. \tag{31}$$

As a corollary, we obtain the pdf from the FPT of a continuous local martingale to a constant boundary if we assume that the quadratic variation $\langle Z\rangle$ is absolutely continuous a.s. Then, there exists derivatives to $\langle Z\rangle = z$, which we define as z'_t for $t \geq 0$ a.e. and a.s.

Corollary 4. *Under Assumption 3 and if we assume that the quadratic variation $\langle Z\rangle$ is absolutely continuous on $\mathbb{R}^+$ a.s., we have that*

$$f^Z_{g,h}(t) \quad = \quad \int_{\mathcal{S}} z'_t f^W_{g_0,h_0}(z_t) dF_u(g_0, h_0, z) \text{ for } t \geq 0 \text{ a.e.} \tag{32}$$

3. Explicit solution of the IFPT problem

In this section, we derive an explicit solution of the IFPT problem for the one-sided and two-sided boundary and in the nonrandom case and random case.

3.1. One-sided nonrandom case

In this part, we consider the case when the one-sided boundary is nonrandom constant and the quadratic variation of the continuous local martingale is a nonrandom time-dependent function.

3.1.1. Case when the quadratic variation is absolutely continuous

To define the IFPT problem, we first introduce the set of cdfs. Since the stochastic process Z is continuous and thus its quadratic variation $\langle Z\rangle$ is also continuous, we accordingly consider the set of continuous cdfs.

Definition 9. A function $F : \mathbb{R}^+ \to [0, 1]$ is a cdf if F is nondecreasing, continuous, satisfies $F(0) = 0$ and $\lim_{t\to\infty} F(t) = 1$.

Since we consider the particular case when the quadratic variation $\langle Z\rangle$ is absolutely continuous, we restrict to the set of absolutely continuous cdfs. Then, we can also define the pdf of F as $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$F(t) \quad = \quad \int_0^t f(s)ds \text{ for } t \geq 0. \tag{33}$$

The IFPT problem determines the increasing function such that the FPT of a standard Wiener process, which is time changed by this increasing function, to the boundary has a given cdf of the form (33). Since we consider increasing functions which are absolutely continuous, we can focus on variance functions.

Definition 10. For a given pdf f, we say that a variance function $\sigma^2_f : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f\rangle_t \quad = \quad \int_0^t \sigma^2_{s,f} ds \text{ for } t \geq 0, \tag{34}$$

is solution if it satisfies

$$P_g^{Z^f}(t) \quad = \quad F(t) \text{ for } t \geq 0. \tag{35}$$

Equation (34) in Definition 10 implicitly requires the existence of a continuous local martingale with quadratic variation $\int_0^t \sigma_{s,f}^2 ds$. This existence can be shown with Itô processes considered in Example 1.

We define the infimum time such that F is positive and the infimum time such that F equals unity as

$$K_F^0 \quad = \quad \inf\{t > 0 \text{ such that } F(t) > 0\} \text{ and} \tag{36}$$

$$K_F^1 \quad = \quad \inf\{t > 0 \text{ such that } F(t) = 1\}. \tag{37}$$

Let us give a set of assumptions sufficient to obtain the explicit solution of the IFPT problem.

Assumption 4. We assume that there exists $\eta_F^0 > 0$ s.t. the explicit solution of the IFPT problem is locally integrable on $[K_F^0, K_F^0 + \eta_F^0]$, i.e.

$$\sigma_f^2 \restriction_{[K_F^0, K_F^0+\eta_F^0]} \in L_{1,loc}\big([K_F^0, K_F^0 + \eta_F^0]\big). \tag{38}$$

Moreover, we assume that K_F^1 is not finite.

We define the error function and its inverse as

$$\operatorname{erf}(t) \quad = \quad \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \text{ for } t \in \mathbb{R}, \tag{39}$$

$$\operatorname{erf}(\operatorname{erfinv}(t)) \quad = \quad t \text{ for } t \in (-1, 1). \tag{40}$$

We also introduce the notation $h(t) = \operatorname{erfinv}(1 - F(t))$. We now give the explicit solution of the IFPT problem. The proof is based on an application of Theorem 1, and the use of elementary topological arguments.

Theorem 5. *Under Assumption 4, the variance function defined as*

$$\sigma_{t,f}^2 = \quad \frac{f(t)}{\frac{2}{g^2\sqrt{\pi}} h(t)^3 e^{-h(t)^2}} \mathbf{1}_{\{0<F(t)<1\}} \quad \textit{for } t \geq 0 \tag{41}$$

is the explicit solution of the IFPT problem.

3.1.2. Case when the quadratic variation is not absolutely continuous

By Equation (11) from Theorem 1, we have that $P_g^Z(t) = P_g^W\big(\langle Z\rangle_t\big)$ for $t \geq 0$. We first give the definition of a solution in the IFPT problem.

Definition 11. For a given cdf F, we say that a nonrandom and nondecreasing function $v_F : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F\rangle_t \quad = \quad v_F(t) \text{ for } t \geq 0, \tag{42}$$

is solution if it satisfies

$$P_g^{Z^F}(t) \quad = \quad F(t) \text{ for } t \geq 0. \tag{43}$$

Equation (42) in Definition 11 implicitly implies the existence of a continuous local martingale with quadratic variation v_F. This is true since a standard Wiener process, which is time-changed by $\frac{v_F(t)}{t}$, will have v_F as quadratic variation.

Let us give an assumption sufficient to obtain the explicit solution of the IFPT problem.

Assumption 5. We assume that K_F^1 is not finite.

We now give the explicit solution of the IFPT problem. The proof is based on an application of Theorem 1 and the use of elementary analysis.

Theorem 6. *Under Assumption 5, the function defined as*

$$v_F(t) = \frac{g^2}{2h(t)^2}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{for } t \geq 0. \tag{44}$$

is the explicit solution of the IFPT problem.

3.2. Two-sided nonrandom case

In this part, we consider the case when the two-sided boundary is nonrandom constant and the quadratic variation of the continuous local martingale is a nonrandom time-dependent function.

3.2.1. Case when the quadratic variation is absolutely continuous

We give the definition of a solution in the IFPT problem.

Definition 12. For a given pdf f, we say that a variance function $\sigma_f^2 : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f \rangle_t = \int_0^t \sigma_{s,f}^2 ds \text{ for } t \geq 0, \tag{45}$$

is solution if it satisfies

$$P_{g,h}^{Z^f}(t) = F(t) \text{ for } t \geq 0. \tag{46}$$

Let us give a set of assumptions sufficient to obtain the explicit solution of the IFPT problem.

Assumption 6. We assume that the explicit solution of the IFPT problem is locally integrable in K_F^0, i.e. there exists $\eta_F^0 > 0$ such that

$$\sigma_f^2 \restriction_{[K_F^0, K_F^0 + \eta_F^0]} \in L_1\big([K_F^0, K_F^0 + \eta_F^0]\big). \tag{47}$$

Moreover, we also assume that K_F^1 is not finite.

We now give the explicit solution of the IFPT problem.

Theorem 7. *Under Assumption 6, the variance function defined as*

$$\sigma_{t,f}^2 = \frac{f(t)}{f_{g,h}^W((P_{g,h}^W)^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{for } t \geq 0 \tag{48}$$

is the explicit solution of the IFPT problem.

3.2.2. *Case when the quadratic variation is not absolutely continuous*

We first give the definition of a solution in the IFPT problem.

Definition 13. For a given cdf F, we say that a nonrandom and nondecreasing function $v_F : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F \rangle_t \quad = \quad v_F(t) \text{ for } t \geq 0, \tag{49}$$

is solution if it satisfies

$$P_{g,h}^{Z^F}(t) \quad = \quad F(t) \text{ for } t \geq 0. \tag{50}$$

We now give the explicit solution of the IFPT problem.

Theorem 8. *Under Assumption 5, the function defined as*

$$v_F(t) = \quad (P_{g,h}^W)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}} \qquad \textit{for } t \geq 0 \tag{51}$$

is the explicit solution of the IFPT problem.

3.3. *One-sided random case*

In this part, we consider the case when the one-sided boundary is random constant and the quadratic variation of the continuous local martingale is a stochastic process.

3.3.1. *Case when the quadratic variation is absolutely continuous*

To define the IFPT problem, we introduce the set of random cdfs. Since the stochastic process Y has its quadratic variation $\langle Y \rangle$ which is continuous and random, we accordingly consider the set of random continuous cdfs.

Definition 14. A function $F : \mathbb{R}^+ \times \Omega \to [0,1]$ is a random cdf if $F(\omega)$ is nondecreasing, continuous, satisfies $F(0,\omega) = 0$ and $\lim_{t\to\infty} F(t,\omega) = 1$ for $\omega \in \Omega$.

Since the quadratic variation $\langle Y \rangle$ is a stochastic process which is absolutely continuous, we restrict to the set of random absolutely continuous cdfs.

Definition 15. A function $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ is a random pdf if it satifies

$$F(t,\omega) \quad = \quad \int_0^t f(s,\omega)ds \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{52}$$

We define the regular conditional cdf of T_g^Z as $P_g^Z(|\omega)$. We give the definition of a solution in the IFPT problem. Since the quadratic variation is a stochastic process which is absolutely continuous, we can focus on variances which are a stochastic process.

Definition 16. For a given random pdf f, we say that a variance process $\sigma_f^2 : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Y^f, i.e.

$$\langle Y^f \rangle_t(\omega) \quad = \quad \int_0^t \sigma_{s,f}^2(\omega) ds \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{53}$$

is solution if it satisfies

$$P_1^{Y^F}(t|\omega) \quad = \quad F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{54}$$

Equation (53) in Definition 16 implicitly requires the existence of a continuous local martingale with random quadratic variation $\int_0^t \sigma_{s,f}^2 ds$. This existence can be shown with Itô processes considered in Example 2.

Let us give a set of assumptions sufficient to obtain the explicit solution of the IFPT problem.

Assumption 7. We assume that the explicit solution is locally integrable on $\mathbb{R}^+ \times \Omega$, i.e.

$$\sigma_f^2 \in L_{1,loc}\big(\mathbb{R}^+ \times \Omega\big). \tag{55}$$

Moreover, we also assume that K_F^1 is not finite.

We introduce the notation $h(t,\omega) = \text{erfinv}(1 - F(t,\omega))$. We now give the explicit solution of the IFPT problem.

Theorem 9. *Under Assumption 7, the variance process defined as*

$$\sigma_{t,f}^2(\omega) = \quad \frac{f(t,\omega)}{\frac{2}{g^2\sqrt{\pi}} h(t,\omega)^3 e^{-h(t,\omega)^2}} \mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \textit{for } t \geq 0 \textit{ and } \omega \in \Omega \tag{56}$$

is the explicit solution of the IFPT problem.

3.3.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of a solution in the IFPT problem.

Definition 17. For a given random cdf F, we say that a nondecreasing stochastic process $v_F : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation of a continuous local martingale Y^F, i.e.

$$\langle Y^F \rangle_t(\omega) \quad = \quad v_F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{57}$$

is solution if it satisfies

$$P_1^{Y^F}(t|\omega) \quad = \quad F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{58}$$

Equation (57) in Definition 17 implicitly implies the existence of a continuous local martingale with quadratic variation v_F. This is true since a standard

Wiener process, which is time-changed by $\frac{v_F(t)}{t}$, will have v_F as quadratic variation.

For $\omega \in \Omega$, we define the infimum time such that $F(t,\omega)$ is positive and the infimum time such that $F(t,\omega)$ equals unity as

$$K_F^0(\omega) = \inf\{t > 0 \text{ such that } F(t,\omega) > 0\} \text{ and} \tag{59}$$

$$K_F^1(\omega) = \inf\{t > 0 \text{ such that } F(t,\omega) = 1\}. \tag{60}$$

Let us give an assumption sufficient to obtain the explicit solution of the IFPT problem.

Assumption 8. We assume that $K_F^1(\omega)$ is not finite for $\omega \in \Omega$.

We now give the explicit solution of the IFPT problem.

Theorem 10. *Under Assumption 8, the stochastic process defined as*

$$v_F(t,\omega) = \tfrac{1}{2h(t,\omega)^2}\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega \tag{61}$$

is the explicit solution of the IFPT problem.

3.4. Two-sided random case

In this part, we consider the case when the two-sided boundary is random constant and the quadratic variation of the continuous local martingale is a stochastic process.

3.4.1. Case when the quadratic variation is absolutely continuous

We define the regular conditional cdf of $\mathrm{T}_{g,h}^Z$ as $P_{g,h}^Z(|\omega)$. We give the definition of a solution in the IFPT problem.

Definition 18. For a given random pdf f, we say that a variance process $\sigma_f^2 : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f \rangle_t(\omega) = \int_0^t \sigma_{s,f}^2(\omega) ds \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{62}$$

is solution if it satisfies

$$P_{g,h}^{Z^F}(t|\omega) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{63}$$

Equation (62) in Definition 18 implicitly requires the existence of a continuous local martingale with random quadratic variation $\int_0^t \sigma_{s,f}^2 ds$. This is true since we can consider Itô processes from Example 2.

We now give the explicit solution of the IFPT problem.

Theorem 11. *Under Assumption 7, the variance process, defined as*

$$\sigma_{t,f}^2(\omega) = \frac{f(t,\omega)}{f_{g,h}^W((P_{g,h}^W)^{-1}(F(t,\omega)))} \mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad (64)$$
$$\textit{for } t \geq 0 \textit{ and } \omega \in \Omega,$$

is the explicit solution of the IFPT problem.

3.4.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of a solution in the IFPT problem.

Definition 19. For a given random cdf F, we say that a nondecreasing stochastic process $v_F : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F \rangle_t(\omega) \quad = \quad v_F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \qquad (65)$$

is solution if it satisfies

$$P_{g,h}^{Z^F}(t|\omega) \quad = \quad F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \qquad (66)$$

Equation (65) in Definition 19 implicitly implies the existence of a continuous local martingale with quadratic variation v_F. This is true since a standard Wiener process, which is time-changed by $\frac{v_F(t)}{t}$, will have v_F as quadratic variation.

We now give the explicit solution of the IFPT problem.

Theorem 12. *Under Assumption 8, the stochastic process defined as*

$$v_F(t,\omega) = \quad (P_{g,h}^W)^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \textit{for } t \geq 0 \textit{ and } \omega \in \Omega \qquad (67)$$

is the explicit solution of the IFPT problem.

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Appendix A: Proofs of the explicit formula

In this section, we prove the explicit formula for the one-sided and two-sided boundary crossing probability (1) and (2) of a continuous local martingale in the nonrandom case and random case.

A.1. One-sided nonrandom case

We start with the proof of Lemma 1, which are well-known results from Malmquist (1954) (Theorem 1, p. 526) and Wang and Pötzelberger (1997) (Equations (3), p. 55).

Proof of Lemma 1. By Malmquist (1954) (Theorem 1, p. 526), we have that the probability that a standard Wiener process crosses a constant boundary g conditioned on its final value x at the final time t is given for $x \in \mathbb{R}$ by

$$\mathbb{P}(\mathrm{T}_g^W \leq t | W_t = x) = \exp\Big(-\frac{2g(g-x)}{t}\Big)\mathbf{1}_{\{x \leq g\}} + \mathbf{1}_{\{x > g\}}. \tag{68}$$

Wang and Pötzelberger (1997) (Equations (3), p. 55) integrate Equation (68) with respect to the Wiener process final value s and derive the cdf as $P_g^W(0) = 0$ and Equation (8). Then, we can deduce the pdf for $t > 0$ as

$$\begin{aligned}
f_g^W(t) &= \frac{d}{dt} P_g^W(t) \\
&= \frac{d}{dt}\left(1 - \Phi\left(\frac{g}{\sqrt{t}}\right) + \Phi\left(\frac{-g}{\sqrt{t}}\right)\right) \\
&= \frac{d}{dt}\left(1 - \int_{-\infty}^{\frac{g}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + \int_{-\infty}^{\frac{-g}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right) \\
&= \frac{g e^{-\frac{g^2}{2t}}}{\sqrt{2\pi t^3}},
\end{aligned}$$

where we use Equation (6) in the first equality, Equation (8) in the second equality, Equation (7) in the third equality, the fundamental theorem of calculus with the chain rule in the fourth equality. We have thus shown Equation (9). □

We define the inverse function of the quadratic variation for $t \geq 0$ and $\omega \in \Omega$ as

$$\langle Z \rangle_t^{-1} = \inf\{s \geq 0 \text{ s.t. } \langle Z \rangle_s > t\}.$$

We also define the canonical filtration of a stochastic process Z as $\mathcal{F}_t^Z = \sigma\big(Z(C), C \in \mathcal{B}(\mathbb{R}^+), C \subset [0,t]\big)$ for $t \geq 0$, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field generated by the open sets of $\mathbb{R}^+$. Finally, we define the process Z, which is time changed by its quadratic variation inverse, as $B_t = Z_{\langle Z \rangle_t^{-1}}$ for $t \geq 0$ and $\omega \in \Omega$. The following lemma states that B is a Wiener process. This is obtained by a direct application of the Dambis, Dubins-Schwarz theorem for continuous local martingale (see Revuz and Yor (2013), Th. V.1.6).

Lemma 3. *Under Assumption 1, we have that B is a $\mathcal{F}^B$-Wiener process and*

$$Z_t = B_{\langle Z\rangle_t} \text{ for } t \geq 0 \text{ a.s.} \tag{69}$$

Proof of Lemma 3. This is obtained by a direct application of the Dambis, Dubins-Schwarz theorem for continuous local martingale (see Revuz and Yor (2013), Th. V.1.6) with Assumption 1. □

We introduce Proposition 1 in what follows. It states that the FPT of Z and B are equal, if we make a time change equal to the quadratic variation of Z. The main elementary idea of the proof is the scale invariant property of the time-changed Wiener process and thus the scale invariant property of the FPT. This is due to the constancy of the boundary.

Proposition 1. *Under Assumption 1, we have for any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z\rangle_t}(\omega)$ that*

$$\left\{\mathrm{T}_g^Z = t\right\} = \left\{\mathrm{T}_g^B = \langle Z\rangle_t\right\} \text{ for } t \geq 0. \tag{70}$$

Proof of Proposition 1. We have for $t \geq 0$ and any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z\rangle_t}(\omega)$ that

$$\begin{aligned} \left\{\mathrm{T}_g^Z = t\right\} &= \left\{\inf\{s \geq 0 \text{ s.t. } Z_s \geq g\} = t\right\} \\ &= \left\{\inf\{s \geq 0 \text{ s.t. } B_{\langle Z\rangle_s} \geq g\} = t\right\}, \end{aligned} \tag{71}$$

where we use Equation (4) in the first equality, and Equation (69) from Lemma 3 with Assumption 1 in the second equality. Since B is a $\mathcal{F}^B$-Wiener process, W is an $\mathcal{F}$-Wiener process and the boundary is constant, we can make a time change equal to the quadratic variation $\langle Z\rangle_t$ and obtain that

$$\left\{\inf\{s \geq 0 \text{ s.t. } B_{\langle Z\rangle_s} \geq g\} = t\right\} = \left\{\inf\{s \geq 0 \text{ s.t. } B_s \geq g\} = \langle Z\rangle_t\right\}. \tag{72}$$

Then, we can calculate by Equation (4) that

$$\left\{\inf\{s \geq 0 \text{ s.t. } B_s \geq g\} = \langle Z\rangle_t\right\} = \left\{\mathrm{T}_g^B = \langle Z\rangle_t\right\}. \tag{73}$$

By Equations (71), (72) and (73), we can deduce Equation (70). □

In what follows, we give the proof of Theorem 1. The proof is mainly based on Proposition 1.

Proof of Theorem 1. We have that for $t \geq 0$

$$\begin{aligned} P_g^Z(t) &= \mathbb{P}(\mathrm{T}_g^Z \leq t) \\ &= \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } Z_s \geq g\} \leq t\big) \\ &= \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{\langle Z\rangle_s} \geq g\} \leq t\big), \end{aligned} \tag{74}$$

where we use Equation (5) in the first equality, Equation (4) in the second equality, and Equation (69) from Lemma 3 with Assumption 1 in the third equality. By Lemma 1 with Assumption 1, we obtain that for $t \geq 0$

$$\mathbb{P}\Big(\inf\{s \geq 0 \text{ s.t. } B_{\langle Z\rangle_s} \geq g\} \leq t\Big) = \mathbb{P}\Big(\inf\{s \geq 0 \text{ s.t. } B_s \geq g\} \leq \langle Z\rangle_t\Big). \tag{75}$$

Then, we can calculate that for $t \geq 0$

$$
\begin{aligned}
\mathbb{P}\Big(\inf\{s \geq 0 \text{ s.t. } B_s \geq g\} \leq \langle Z\rangle_t\Big) &= \mathbb{P}\Big(\inf\{s \geq 0 \text{ s.t. } W_s \geq g\} \leq \langle Z\rangle_t\Big) \\
&= \mathbb{P}\Big(\mathrm{T}_g^W \leq \langle Z\rangle_t\Big), \\
&= P_g^W\big(\langle Z\rangle_t\big). \qquad (76)
\end{aligned}
$$

Here, we use the fact that B and W have the same distribution in the first equality, Equation (4) in the second equality, and Equation (5) in the third equality. By Equations (74), (75) and (76), we can deduce Equation (11). □

Finally, we give the proof of Corollary 1.

Proof of Corollary 1. We have for $t \geq 0$ a.e.

$$
\begin{aligned}
f_g^X(t) &= \frac{dP_g^X(t)}{dt} \\
&= \frac{d(P_g^W(\langle Z\rangle_t))}{dt} \\
&= \langle Z\rangle_t' f_g^W\big(\langle Z\rangle_t\big).
\end{aligned}
$$

where we use Equation (6) in the first equality, Equation (11) from Theorem 1 with Assumption 1 in the second equality, and the fundamental theorem of calculus with the chain rule and the assumption that the quadratic variation $\langle Z\rangle$ is absolutely continuous on $\mathbb{R}^+$ in the third equality. □

A.2. Two-sided nonrandom case

We start with the proof of Lemma 2, which is well-known results from Anderson (1960) (Theorem 4.3, p. 180).

Proof of Lemma 2. Equation (18) is a more compact form of Theorem 4.3 (p. 180) in Anderson (1960). Then, we derive the integral of $\mathrm{ss}(v,w)$ for $0 < v < w$ as

$$
\begin{aligned}
\int_0^t \mathrm{ss}_x(v,w)dx &= \sum_{k=-\infty}^{\infty} \frac{w-v+2kw}{\sqrt{2\pi}} \int_0^t x^{-3/2} e^{-(w-v+2kw)^2/2x} dx \\
&= \sum_{k=-\infty}^{\infty} \left(2 - 2\Phi\left(\frac{w-v+2kw}{\sqrt{t}}\right)\right), \qquad (77)
\end{aligned}
$$

where we use Equation (16) in the first equality. Then, we can obtain $P_{g,h}^W(0) = 0$ and Equation (17) by integrating Equation (18) with the use of Equation (77) for $t > 0$. □

We introduce Proposition 2 in what follows. It states that the FPT of Z and B are equal, if we make a time change equal to the quadratic variation of Z.

Proposition 2. *Under Assumption 1, we have for any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z \rangle_t}(\omega)$ that*

$$\big\{ \mathrm{T}^Z_{g,h} = t \big\} \quad = \quad \big\{ \mathrm{T}^B_{g,h} = \langle Z \rangle_t \big\} \text{ for } t \geq 0. \tag{78}$$

Proof of Proposition 2. We have for $t \geq 0$ and any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z \rangle_t}(\omega)$ that

$$\begin{aligned} \big\{ \mathrm{T}^Z_{g,h} = t \big\} &= \big\{ \inf\{ s \geq 0 \text{ s.t. } Z_s \geq g \text{ or } Z_s \leq h \} = t \big\} \\ &= \big\{ \inf\{ s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h \} = t \big\}, \end{aligned} \tag{79}$$

where we use Equation (13) in the first equality, and Equation (69) from Lemma 3 with Assumption 1 in the second equality. Since B is a $\mathcal{F}^B$-Wiener process, W is an $\mathcal{F}$-Wiener process and the boundary is constant, we can make a time change equal to the quadratic variation $\langle Z \rangle_t$ and obtain that

$$\begin{aligned} &\big\{ \inf\{ s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h \} = t \big\} \\ &= \big\{ \inf\{ s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h \} = \langle Z \rangle_t \big\}. \end{aligned} \tag{80}$$

Then, we can calculate by Equation (13) that

$$\big\{ \inf\{ s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h \} = \langle Z \rangle_t \big\} \quad = \quad \big\{ \mathrm{T}^B_{g,h} = \langle Z \rangle_t \big\}. \tag{81}$$

By Equations (79), (80) and (81), we can deduce Equation (78). □

In what follows, we give the proof of Theorem 2.

Proof of Theorem 2. We have that for $t \geq 0$

$$\begin{aligned} P^Z_{g,h}(t) &= \mathbb{P}(\mathrm{T}^Z_{g,h} \leq t) \\ &= \mathbb{P}\big(\inf\{ s \geq 0 \text{ s.t. } Z_s \geq g \text{ or } Z_s \leq h \} \leq t \big) \\ &= \mathbb{P}\big(\inf\{ s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h \} \leq t \big), \end{aligned} \tag{82}$$

where we use Equation (14) in the first equality, Equation (13) in the second equality, and Equation (69) from Lemma 3 with Assumption 1 in the third equality. By Lemma 2 with Assumption 1, we obtain that

$$\begin{aligned} &\mathbb{P}\Big(\inf\{ s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h \} \leq t \Big) \\ &= \mathbb{P}\Big(\inf\{ s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h \} \leq \langle Z \rangle_t \Big). \end{aligned} \tag{83}$$

Then, we can calculate that

$$\begin{aligned} &\mathbb{P}\Big(\inf\{ s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h \} \leq \langle Z \rangle_t \Big) \\ = \; &\mathbb{P}\Big(\inf\{ s \geq 0 \text{ s.t. } W_s \geq g \text{ or } W_s \leq h \} \leq \langle Z \rangle_t \Big) \\ = \; &\mathbb{P}\Big(\mathrm{T}^W_{g,h} \leq \langle Z \rangle_t \Big), \\ = \; &P^W_{g,h}\big(\langle Z \rangle_t \big), \end{aligned} \tag{84}$$

where we use the fact that B and W have the same distribution in the second equality, Equation (13) in the third equality, and Equation (14) in the fourth equality. By Equations (82), (83) and (84), we can deduce Equation (19). $\square$

Finally, we give the proof of Corollary 2.

Proof of Corollary 2. We have for $t \geq 0$ a.e.

$$\begin{aligned} f_{g,h}^X(t) &= \frac{dP_{g,h}^X(t)}{dt} \\ &= \frac{d(P_{g,h}^W(\langle Z \rangle_t))}{dt} \\ &= \langle Z \rangle_t' f_{g,h}^W(\langle Z \rangle_t), \end{aligned}$$

where we use Equation (15) in the first equality, Equation (19) from Theorem 2 with Assumption 1 in the second equality, and the fundamental theorem of calculus with the chain rule and the assumption that the quadratic variation $\langle Z \rangle$ is absolutely continuous on $\mathbb{R}^+$ in the third equality. $\square$

A.3. One-sided random case

We define the inverse function of the quadratic variation for $t \geq 0$ and $\omega \in \Omega$ as

$$\langle Y \rangle_t^{-1} = \inf\{s \geq 0 \text{ s.t. } \langle Y \rangle_s > t\}.$$

Finally, we define the process Y, which is time changed by its quadratic variation inverse, as $B_t = Y_{\langle Y \rangle_t^{-1}}$ for $t \geq 0$ and $\omega \in \Omega$. The following lemma states that B is a Wiener process.

Lemma 4. *Under Assumption 2, we have that B is a $\mathcal{F}^B$-Wiener process and*

$$Y_t = B_{\langle Y \rangle_t} \text{ for } t \geq 0 \text{ a.s.}. \tag{85}$$

Proof of Lemma 4. This is obtained by a direct application of the Dambis, Dubins-Schwarz theorem for continuous local martingale (see Revuz and Yor (2013), Th. V.1.6) with Assumption 2. $\square$

We introduce Proposition 3 in what follows. The main elementary idea of the proof is the scale invariant property of the time-changed Wiener process and thus the scale invariant property of the FPT which adapts to the one-sided random case by using the new process.

Proposition 3. *Under Assumption 2, we have for any $\omega \in \Omega$ satisfying $Y_t(\omega) = B_{\langle Y \rangle_t}(\omega)$ that*

$$\left\{\mathrm{T}_1^Y = t\right\} = \left\{\mathrm{T}_1^B = \langle Y \rangle_t\right\} \text{ for } t \geq 0. \tag{86}$$

Proof of Proposition 3. We have that for $t \geq 0$ and any $\omega \in \Omega$ satisfying $Y_t(\omega) = B_{\langle Y \rangle_t}(\omega)$

$$\begin{aligned} \big\{ \mathrm{T}_1^Y = t \big\} &= \big\{ \inf\{s \geq 0 \text{ s.t. } Y_s \geq 1\} = t \big\} \\ &= \big\{ \inf\{s \geq 0 \text{ s.t. } B_{\langle Y \rangle_s} \geq 1\} = t \big\}, \end{aligned} \tag{87}$$

where we use Equation (21) in the first equality, and Equation (85) from Lemma 4 with Assumption 2 in the second equality. Since B is a $\mathcal{F}^B$-Wiener process, W is an $\mathcal{F}$-Wiener process and the boundary is constant, we can make a time change equal to the quadratic variation $\langle Y \rangle_t$ and obtain that

$$\begin{aligned} &\big\{ \inf\{s \geq 0 \text{ s.t. } B_{\langle Y \rangle_s} \geq 1\} = t \big\} \\ &= \big\{ \inf\{s \geq 0 \text{ s.t. } B_s \geq 1\} = \langle Y \rangle_t \big\}. \end{aligned} \tag{88}$$

Then, we can calculate by Equation (21) that

$$\big\{ \inf\{s \geq 0 \text{ s.t. } B_s \geq 1\} = \langle Y \rangle_t \big\} = \big\{ \mathrm{T}_1^B = \langle Y \rangle_t \big\}. \tag{89}$$

By Equations (87), (88) and (89), we can deduce Equation (86). □

In what follows, we give the proof of Theorem 3. The proof is mainly based on Proposition 3. We get P_1^Y in the proof of Theorem 3 by regular conditional probability, and using the explicit formula obtained in the nonrandom case.

Proof of Theorem 3. We have that for $t \geq 0$

$$\begin{aligned} P_1^Y(t) &= \mathbb{P}(\mathrm{T}_1^Y \leq t) \\ &= \int_{\mathcal{P}} \mathbb{P}(\mathrm{T}_1^Y \leq t | \langle Y \rangle = y) dF_{\langle Y \rangle}(y) \\ &= \int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } Y_s \geq 1\} \leq t | \langle Y \rangle = y \big) dF_{\langle Y \rangle}(y) \\ &= \int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{y_s} \geq 1\} \leq t \big) dF_{\langle Y \rangle}(y), \end{aligned} \tag{90}$$

where we use Equation (22) in the first equality, regular conditional probability in the second equality, Equation (21) in the third equality, and Equation (85) in the fourth equality. By Lemma 3 with Assumption 2, we obtain that

$$\begin{aligned} &\int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{y_s} \geq 1\} \leq t \big) dF_{\langle Y \rangle}(y) \\ &= \int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_s \geq 1\} \leq y_t \big) dF_{\langle Y \rangle}(y). \end{aligned} \tag{91}$$

Then, we can calculate that

$$
\begin{aligned}
& \int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_s \geq 1\} \leq y_t\big) dF_{\langle Y \rangle}(y) \\
= & \int_{\mathcal{P}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } W_s \geq 1\} \leq y_t\big) dF_{\langle Y \rangle}(y) \\
= & \int_{\mathcal{P}} \mathbb{P}\big(\mathrm{T}_1^W \leq y_t\big) dF_{\langle Y \rangle}(y) \\
= & \int_{\mathcal{P}} P_1^W(y_t) dF_{\langle Y \rangle}(y), \qquad (92)
\end{aligned}
$$

where we use the fact that B and W have the same distribution in the first equality, Equation (21) in the second equality, and Equation (22) in the third equality. By Equations (90), (91) and (92), we can deduce Equation (26). $\square$

Finally, we give the proof of Corollary 3.

Proof of Corollary 3. We have for $t \geq 0$ a.e.

$$
\begin{aligned}
f_1^Y(t) &= \frac{dP_1^Y(t)}{dt} \\
&= \frac{d(\int_{\mathcal{P}} P_1^W(y_t) dF_{\langle Y \rangle}(y))}{dt} \\
&= \int_{\mathcal{P}} \frac{d(P_1^W(y_t))}{dt} dF_{\langle Y \rangle}(y) \\
&= \int_{\mathcal{P}} y_t' f_1^W(y_t) dF_{\langle Y \rangle}(y),
\end{aligned}
$$

where we use Equation (23) in the first equality, Equation (26) from Theorem 3 with Assumption 2 in the second equality, Tonelli's theorem in the third equality, and the fundamental theorem of calculus with chain rule and the assumption that the quadratic variation $\langle Y \rangle$ is absolutely continuous on $\mathbb{R}^+$ a.s. in the fourth equality. $\square$

A.4. Two-sided random case

We define the inverse function of the quadratic variation for $t \geq 0$ and $\omega \in \Omega$ as

$$\langle Z \rangle_t^{-1} = \inf\{s \geq 0 \text{ s.t. } \langle Z \rangle_s > t\}.$$

Finally, we define the process Z, which is time changed by its quadratic variation inverse, as $B_t = Z_{\langle Z \rangle_t^{-1}}$ for $t \geq 0$ and $\omega \in \Omega$. The following lemma states that B is a Wiener process.

Lemma 5. *Under Assumption 3, we have that B is a $\mathcal{F}^B$-Wiener process and*

$$Z_t = B_{\langle Z \rangle_t} \text{ for } t \geq 0 \text{ a.s.}. \qquad (93)$$

Proof of Lemma 5. This is obtained by a direct application of the Dambis, Dubins-Schwarz theorem for continuous local martingale (see Revuz and Yor (2013), Th. V.1.6) with Assumption 3. $\square$

We introduce Proposition 4 in what follows.

Proposition 4. *Under Assumption 3, we have for any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z \rangle_t}(\omega)$ that*

$$\left\{\mathrm{T}^Z_{g,h} = t\right\} \quad = \quad \left\{\mathrm{T}^B_{g,h} = \langle Z \rangle_t\right\} \text{ for } t \geq 0. \tag{94}$$

Proof of Proposition 4. We have for $t \geq 0$ and any $\omega \in \Omega$ satisfying $Z_t(\omega) = B_{\langle Z \rangle_t}(\omega)$ that

$$\begin{aligned} \left\{\mathrm{T}^Z_{g,h} = t\right\} &= \left\{\inf\{s \geq 0 \text{ s.t. } Z_s \geq g \text{ or } Z_s \leq h\} = t\right\} \\ &= \left\{\inf\{s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h\} = t\right\}, \end{aligned} \tag{95}$$

where we use Equation (28) in the first equality, and Equation (93) from Lemma 5 with Assumption 3 in the second equality. Since B is a $\mathcal{F}^B$-Wiener process, W is an $\mathcal{F}$-Wiener process and the boundary is constant, we can make a time change equal to the quadratic variation $\langle Z \rangle_t$ and obtain that

$$\begin{aligned} &\left\{\inf\{s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h\} = t\right\} \\ &= \left\{\inf\{s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h\} = \langle Z \rangle_t\right\}. \end{aligned} \tag{96}$$

Then, we can calculate by Equation (28) that

$$\left\{\inf\{s \geq 0 \text{ s.t. } B_s \geq g \text{ or } B_s \leq h\} = \langle Z \rangle_t\right\} \quad = \quad \left\{\mathrm{T}^B_{g,h} = \langle Z \rangle_t\right\}. \tag{97}$$

By Equations (95), (96) and (97), we can deduce Equation (94). $\square$

In what follows, we give the proof of Theorem 4. The proof is mainly based on Proposition 4. We get $P^Z_{g,h}$ in the next theorem by regular conditional probability, and using the explicit formula obtained in the nonrandom case.

Proof of Theorem 4. We have that for $t \geq 0$

$$\begin{aligned} P^Z_{g,h}(t) &= \mathbb{P}(\mathrm{T}^Z_{g,h} \leq t) \\ &= \int_{\mathcal{S}} \mathbb{P}(\mathrm{T}^Z_{g,h} \leq t | u = (g_0, h_0, z)) dF_u(g_0, h_0, z) \\ &= \int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } Z_s \geq g \text{ or } Z_s \leq h\} \leq t | u = (g_0, h_0, z)) \\ &\quad dF_u(g_0, h_0, z) \\ &= \int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{\langle Z \rangle_s} \geq g \text{ or } B_{\langle Z \rangle_s} \leq h\} \leq t \\ &\quad | u = (g_0, h_0, z)) dF_u(g_0, h_0, z), \end{aligned} \tag{98}$$

where we use Equation (29) in the first equality, regular conditional probability in the second equality, Equation (28) in the third equality, and Equation (93) from Lemma 5 with Assumption 3 in the fourth equality. Since the stochastic process Z is independent from the two-sided boundary (g,h), we obtain that

$$\begin{aligned} &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{\langle Z\rangle_s} \geq g \text{ or } B_{\langle Z\rangle_s} \leq h\} \leq t | u = (g_0,h_0,z)\big) \\ &dF_u(g_0,h_0,z) \\ = &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{z_s} \geq g_0 \text{ or } B_{z_s} \leq h_0\} \leq t\big) dF_u(g_0,h_0,z). \end{aligned} \tag{99}$$

By Lemma 4 with Assumption 3, we obtain that

$$\begin{aligned} &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_{z_s} \geq g_0 \text{ or } B_{z_s} \leq h_0\} \leq t\big) dF_u(g_0,h_0,z) \\ = &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_s \geq g_0 \text{ or } B_s \leq h_0\} \leq z_t\big) dF_u(g_0,h_0,z). \end{aligned} \tag{100}$$

Then, we can calculate that

$$\begin{aligned} &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } B_s \geq g_0 \text{ or } B_s \leq h_0\} \leq z_t\big) dF_u(g_0,h_0,z) \\ = &\int_{\mathcal{S}} \mathbb{P}\big(\inf\{s \geq 0 \text{ s.t. } W_s \geq g_0 \text{ or } W_s \leq h_0\} \leq z_t\big) dF_u(g_0,h_0,z) \\ = &\int_{\mathcal{S}} \mathbb{P}\big(\mathrm{T}^W_{g_0,h_0} \leq z_t\big) dF_u(g_0,h_0,z) \\ = &\int_{\mathcal{S}} P^W_{g_0,h_0}\big(z_t\big) dF_u(g_0,h_0,z), \end{aligned} \tag{101}$$

where we use the fact that B and W have the same distribution in the first equality, Equation (28) in the second equality, and Equation (29) in the third equality. By Equations (98), (99),(100) and (101), we can deduce Equation (31). □

Finally, we give the proof of Corollary 4.

Proof of Corollary 4. We have for $t \geq 0$ a.e.

$$\begin{aligned} f^Z_{g,h}(t) &= \frac{dP^Z_{g,h}(t)}{dt} \\ &= \frac{d\big(\int_{\mathcal{S}} P^W_{g_0,h_0}\big(z_t\big) dF_u(g_0,h_0,z)\big)}{dt} \\ &= \int_{\mathcal{S}} \frac{d\big(P^W_{g_0,h_0}\big(z_t\big)\big)}{dt} dF_u(g_0,h_0,z) \\ &= \int_{\mathcal{S}} z'_t f^W_{g_0,h_0}\big(z_t\big) dF_u(g_0,h_0,z), \end{aligned}$$

where we use Equation (30) in the first equality, Equation (31) from Theorem 4 with Assumption 3 and the assumption that the stochastic process Z is independent from the two-sided boundary (g,h) in the second equality, Tonelli's theorem in the third equality, and the fundamental theorem of calculus with chain rule and the assumption that the quadratic variation $\langle Z\rangle$ is absolutely continuous on $\mathbb{R}^+$ a.s. in the fourth equality. $\square$

Appendix B: Proofs of the explicit solution in the IFPT problem

In this section, we prove the explicit solution of the IFPT problem for the one-sided and two-sided boundary in the case (i) and (ii).

B.1. One-sided nonrandom case

B.1.1. Case when the quadratic variation is absolutely continuous

When P_g^W is invertible, we define its inverse as $(P_g^W)^{-1} : [0,1) \to \mathbb{R}^+$. The first lemma shows that there exists an inverse of P_g^W, and gives explicit formulae of $(P_g^W)^{-1}(t)$ and $f_g^W((P_g^W)^{-1}(t))$ for $0 \le t < 1$, all of which are new results which will be useful to express the explicit solution of the IFPT problem. The proof relies on Lemma 1.

Lemma 6. *There exists an inverse of P_g^W which is strictly increasing such that $(P_g^W)^{-1}(0) = 0$ and*

$$(P_g^W)^{-1}(t) \quad = \quad \frac{g^2}{2\,\mathrm{erfinv}(1-t)^2} \text{ for } 0 < t < 1. \tag{102}$$

Finally, we have $f_g^W((P_g^W)^{-1}(t)) = 0$ and

$$f_g^W((P_g^W)^{-1}(t)) = \frac{2}{g^2\sqrt{\pi}}\mathrm{erfinv}(1-t)^3 e^{-\mathrm{erfinv}(1-t)^2} \text{ for } 0 < t < 1. \tag{103}$$

Proof of Lemma 6. Using Equation (8) from Lemma 1, Equation (7) with Equation (39), we can express the relation between the cdf of the standard normal and the error function as

$$\phi(x) = \frac{1}{2}\Big(1 + \mathrm{erf}(\frac{x}{\sqrt{2}})\Big). \tag{104}$$

We can rewrite Equation (8) as

$$\begin{aligned} P_g^W(t) &= 1 - \frac{1}{2}\Big(1 + \mathrm{erf}(\frac{g}{\sqrt{2t}})\Big) + \frac{1}{2}\Big(1 + \mathrm{erf}(\frac{-g}{\sqrt{2t}})\Big) \\ &= 1 - \frac{1}{2}\Big(1 + \mathrm{erf}(\frac{g}{\sqrt{2t}})\Big) + \frac{1}{2}\Big(1 - \mathrm{erf}(\frac{g}{\sqrt{2t}})\Big) \\ &= 1 - \mathrm{erf}(\frac{g}{\sqrt{2t}}), \end{aligned}$$

We note that $P_g^W : \mathbb{R}^+ \to [0,1)$ is strictly increasing since $f_g^W(t) > 0$ for $t > 0$ by Equation (9). Thus, there exists an inverse $(P_g^W)^{-1} : [0,1) \to \mathbb{R}^+$ which is strictly increasing. First, note that as $P_g^W(0) = 0$, this implies that $(P_g^W)^{-1}(0) = 0$. Using Equation (8), some algebraic manipulation leads to Equation (102). Finally, applying Equation (9) yields the form of $f_g^W((P_g^{W^{-1}}(t))$, i.e., Equation (103). ☐

We then give a lemma whose proof relies on Lemma 1 and Lemma 6. For $A \subset \mathbb{R}^+$ and $B \subset \mathbb{R}^+$, we denote the space $\mathcal{C}_1$ of functions $k : A \to B$ with derivatives which are continuous as $\mathcal{C}_1(A, B)$.

Lemma 7. *We have*

$$f_g^W \in \mathcal{C}_1(\mathbb{R}^+, \mathbb{R}^+) \text{ and } P_g^W \in \mathcal{C}_1(\mathbb{R}^+, [0, 1)), \tag{105}$$

$$(P_g^W)^{-1} \in \mathcal{C}_1([0, 1), \mathbb{R}^+). \tag{106}$$

Proof of Lemma 7. By Equations (8) and (9) in Lemma 1, we obtain Equation (105). By Equation (102) in Lemma 6, we obtain Equation (106). ☐

We now give the definition of an explicit solution.

Definition 20. For a given pdf f, we say that a variance function $\sigma_f^2 : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f \rangle_t = \int_0^t \sigma_{s,f}^2 ds \text{ for } t \geq 0, \tag{107}$$

is an explicit solution if it is of the form

$$\sigma_{t,f}^2 = \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))} \mathbf{1}_{\{0<F(t)<1\}} \qquad \text{for } t \geq 0. \tag{108}$$

If we substitute $(P_g^W)^{-1}$ in Equation (108) with Equation (102) from Lemma 6, we can reexpress the explicit solution as Equation (41). The next proposition shows that Assumption 4 implies that Z^f satisfies Assumption 1. The proof is mainly based on elementary topological arguments in $\mathbb{R}^+$.

Proposition 5. *Under Assumption 4, we have that Z^f satisfies Assumption 1.*

Proof of Proposition 5. To prove that Z^f satisfies Assumption 1, we first show that $\sigma_f^2 \in L_{1,\text{loc}}(\mathbb{R}^+)$, i.e., we have to show by definition that $\forall K \subset \mathbb{R}^+$, K compact, we have

$$\int_K \sigma_{t,f}^2 \, \mathrm{d}t < +\infty. \tag{109}$$

There is no loss of generality assuming that K has a closed interval form $K = [K_0, K_1]$ where $0 \leq K_0 < K_1$, since if not we can break K into a finite number of nonoverlapping closed intervals by the Bolzano-Weierstrass theorem and prove Equation (109) for each interval. We first consider the case where

$$0 \leq K_F^0 < K_0 < K_1. \tag{110}$$

Given the form of the explicit solution (108), Equation (109) can be reexpressed as

$$\int_K \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))} \,\mathrm{d}t < +\infty. \tag{111}$$

We first show that the denominator in the integral of Equation (111) is uniformly bounded away from 0 on K. By Definition 9, F is a cdf and thus a nondecreasing function. We can deduce that

$$F(K_0) \leq F(t) \leq F(K_1) \text{ for } t \in K. \tag{112}$$

We also obtain by definition of K_F^0 in Equation (36), definition of K_F^1 in Equation (37) and the assumption that K_F^1 is not finite from Assumption 4 that

$$0 < F(\widetilde{K}) < 1 \text{ for } \widetilde{K} \in \mathbb{R} \text{ such that } K_F^0 < \widetilde{K}. \tag{113}$$

Combining Equations (112) and (113), we can deduce that

$$0 < F(K_0) \leq F(t) \leq F(K_1) < 1, \text{ for } t \in K. \tag{114}$$

By Lemma 6, we have that $(P_g^W)^{-1}$ is strictly increasing. Thus, applying $(P_g^W)^{-1}$ to each term of Inequality (114) yields

$$0 < (P_g^W)^{-1}(F(K_0)) \leq (P_g^W)^{-1}(F(t))) \leq (P_g^W)^{-1}(F(K_1)), \text{ for } t \in K. \tag{115}$$

We have that $(P_g^W)^{-1}(F(t)))$ takes its values in the closed interval

$$[(P_g^W)^{-1}(F(K_0)), (P_g^W)^{-1}(F(K_1))]$$

of $\mathbb{R}^+$ which is connected and compact by the Bolzano-Weierstrass theorem. Besides, it is known from topological properties that the image of a compact and connected set of $\mathbb{R}^+$ by a continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ is a compact and connected set of $\mathbb{R}^+$. Since f_g^W is continuous by Equation (9), we can deduce that $f_g^W((P_g^W)^{-1}(F(t)))$ for $t \in K$ is included into a compact and connected space of $\mathbb{R}^+$, e.g., a closed interval of $\mathbb{R}^+$. From Equation (9), we get that there exists $C > 0$ such that

$$C \leq f_g^W((P_g^W)^{-1}(F(t))) \text{ for } t \in K. \tag{116}$$

This implies that the denominator in the integral of Equation (111) is uniformly bounded away from 0 on K. Given that f is a pdf, we obtain that

$$\int_K f(t) \,\mathrm{d}t < +\infty.$$

Thus, Equation (111) holds. We now consider the general case when K is not necessarily of the form (110). We consider the case when $K_0 \leq K_F^0 < K_1$. If we introduce the notation $\widetilde{K}_F^0 = K_F^0 + \eta_F^0$, then we can decompose $[K_0, K_1]$ as

$$[K_0, K_1] = [K_0, K_F^0] \cup [K_F^0, \widetilde{K}_F^0] \cup [\widetilde{K}_F^0, K_1].$$

We deduce that

$$\begin{aligned}\int_K \sigma^2_{t,f}\,\mathrm{d}t &= \int_{[K_0,K_F^0]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[K_F^0,\widetilde{K}_F^0]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[\widetilde{K}_F^0,\widetilde{K}_F^1]} \sigma^2_{t,f}\,\mathrm{d}t \\ &= \int_{[K_F^0,\widetilde{K}_F^0]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[\widetilde{K}_F^0,\widetilde{K}_F^1]} \sigma^2_{t,f}\,\mathrm{d}t \\ &\leq C + \int_{[\widetilde{K}_F^0,\widetilde{K}_F^1]} \sigma^2_{t,f}\,\mathrm{d}t \\ &< +\infty,\end{aligned}$$

where the second equality is due to the fact that the variance function is null on $[K_0, K_F^0]$ by Equation (108), the first inequality with $C > 0$ follows by Expression (38) from Assumption 4, and the second inequality is due to Equation (111). Finally, we have that the variance function is null on by Equation (108) in the case when $K_F^0 \leq K_0$. We have thus shown Expression (109). Thus, we can deduce that Z^f is a local martingale with nonrandom quadratic variation

$$\langle Z^f\rangle_t = \int_0^t \sigma^2_{u,f}du \tag{117}$$

by Theorem I.4.40 (p. 48) from Jacod and Shiryaev (2003) with Expression (109). Finally, we show that $\langle Z^f\rangle_t \to \infty$ as $t \to \infty$. We can calculate that

$$\begin{aligned}\langle Z^f\rangle_t &= \langle Z^F\rangle_t \\ &= v_F(t) \\ &= \frac{g^2}{2\,\mathrm{erfinv}(1-F(t))^2}\mathbf{1}_{\{0<F(t)<1\}},\end{aligned} \tag{118}$$

where we use the fact that $Z^f = Z^F$ in the first equality, Equation (124) from Definition 21 in the second equality, and Equation (44) in the last equality. By definition we have that $\mathrm{erfinv}(z) \to 0$ as $z \to 0$, and by Definition 9 we have that $\lim_{t\to\infty} F(t) = 1$. Thus, we can deduce by the assumption that K_F^1 is finite from Assumption 4 that

$$\frac{g^2}{2\,\mathrm{erfinv}(1-F(t))^2}\mathbf{1}_{\{0<F(t)<1\}} \to 0 \tag{119}$$

as $t \to \infty$. We can deduce by Equations (117), (118) and (119) that $\langle Z^f\rangle_t \to \infty$ as $t \to \infty$. This implies that Z^f satisfies Assumption 1.

□

The next proposition states that if Z^f satisfies Assumption 1, then, the variance function is a solution if and only if it is an explicit solution. The proof is based on an application of Theorem 1, and elementary analysis. More specifically, it is based on substituting the left-hand side of Equation (35) with Equation (11) from Theorem 1 and Equation (107), and then differentiating and inverting on both sides of the equation to derive the explicit solution.

Proposition 6. *If we assume that Z^f satisfies Assumption 1, then, (i) σ_f^2 is a solution of Definition 10 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 20.*

Proof of Proposition 6. Proof of $(i) \implies (ii)$. We assume that σ_f^2 is a solution of Definition 10. Given that Z^f satisfies Assumption 1, we can substitute the left-hand side of Equation (35) with Equation (11) to deduce

$$P_g^W\Big(\langle Z^f\rangle_t\Big) = F(t) \text{ for } t \geq 0. \tag{120}$$

Using Equation (107), Equation (120) can be reexpressed as

$$P_g^W\Big(\int_0^t \sigma_{s,f}^2 ds\Big) = F(t) \text{ for } t \geq 0. \tag{121}$$

By Lemma 6, there exists an inverse$(P_g^W)^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P_g^W)^{-1}$ on both sides of Equation (121), Equation (121) can be rewritten as

$$\int_0^t \sigma_{s,f}^2 ds = \quad (P_g^W)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}} \quad \text{ for } t \geq 0. \tag{122}$$

The left-hand side of Equation (122) and F have a derivative a.e. for $t \geq 0$ by absolute continuity properties and since F is absolutely continuous. $(P_g^W)^{-1}$ is differentiable on $[0,1)$ by Lemma 7. Thus, we can differentiate (122) a.e. on both sides, by using the chain rule on the right-hand side. We obtain

$$\sigma_{t,f}^2 = \quad f(t)((P_g^W)^{-1})'(F(t))\mathbf{1}_{\{0<F(t)<1\}} \quad \text{a.e. for } t \geq 0. \tag{123}$$

Applying the inverse function theorem, Equation (123) can be reexpressed as

$$\sigma_{t,f}^2 = \quad \tfrac{f(t)}{(P_g^W)'((P_g^W)^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{a.e. for } t \geq 0,$$

or equivalently of the form (108) as $(P_g^W)'(t) = f_g(t)$ for $t \geq 0$ a.e.. Thus, we have shown that σ_f^2 is an explicit solution of Definition 20.

Proof of $(ii) \implies (i)$. We assume that σ_f^2 is an explicit solution of Definition 20. We have for $t \geq 0$

$$\begin{aligned}
P_g^{Z^f}(t) &= P_g^W(\int_0^t \sigma_{s,f}^2 ds) \\
&= P_g^W(\int_0^t \frac{f(s)}{f_g^W((P_g^W)^{-1}(F(s)))}\mathbf{1}_{\{0<F(t)<1\}} ds) \\
&= P_g^W(\int_0^t f(s)((P_g^W)^{-1})'(F(s))\mathbf{1}_{\{0<F(t)<1\}} ds) \\
&= P_g^W((P_g^W)^{-1})(F(t)) \\
&= F(t),
\end{aligned}$$

where we use Equation (11) with the assumption that Z^f satisfies Assumption 1 in the first equality, Equation (108) in the second equality, the inverse function theorem in the third equality, integration in the fourth equality and algebraic manipulation in the fifth equality. We have thus shown that σ_f^2 satisfies Equation (34), and thus that σ_f^2 is a solution of Definition 10. $\square$

The following theorem states that under Assumption 4, (a) Z^f satisfies Assumption 1 and (b) that variance function is solution if and only if it is an explicit solution. The proof of Theorem 13 is a direct application of Proposition 5 and Proposition 6.

Theorem 13. *Under Assumption 4, (a) Z^f satisfies Assumption 1 (b) (i) σ_f^2 is a solution of Definition 10 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 20.*

Proof of Theorem 13. To obtain (a), we apply Proposition 5 with Assumption 4. Then, an application of Proposition 6 with (a) yields (b). $\square$

Finally, we give the proof of Theorem 5, which is a direct consequence of Theorem 13.

Proof of Theorem 5. This is a direct consequence of Theorem 13 with Assumption 4. $\square$

B.1.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of the explicit solution.

Definition 21. For a given cdf F, we say that a nonrandom nondecreasing function v_F which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F \rangle_t \quad = \quad v_F(t) \text{ for } t \geq 0, \tag{124}$$

is an explicit solution if it is of the form

$$v_F(t) = \quad (P_g^W)^{-1}(F(t)) \mathbf{1}_{\{0<F(t)<1\}} \qquad \text{for } t \geq 0. \tag{125}$$

If we substitute $(P_g^W)^{-1}$ in Equation (125) with Equation (102) from Lemma 6, we can reexpress the explicit solution as Equation (44).

The next proposition shows that Assumption 5 implies that Z^F satisfies Assumption 1.

Proposition 7. *Under Assumption 5, we have that Z^F satisfies Assumption 1.*

Proof of Proposition 7. By Definition 21, Z^F is defined as a continuous local martingale with quadratic variation $\langle Z^F \rangle_t = v_F(t)$ for $t \geq 0$, which can be expressed by Equation (44) as

$$v_F(t) = \quad \tfrac{g^2}{2\,\mathrm{erfinv}(1-F(t))^2} \mathbf{1}_{\{0<F(t)<1\}} \qquad \text{for } t \geq 0.$$

By definition we have that $\mathrm{erfinv}(t) \to 0$ as $t \to 0$, and by Definition 9 we have that $\lim\limits_{t\to\infty} F(t) = 1$. Thus, we can deduce by Assumption 5 that $\lim\limits_{t\to\infty} v_F(t) = \infty$. This implies that $\langle Z^F \rangle_\infty = \infty$ and thus that Z^F satisfies Assumption 1. $\square$

The next proposition states that if a nondecreasing function satisfies Assumption 1, then it is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (43) with Equation (11) from Theorem 1 and Equation (124), and then inverting on both sides of the equation to derive the explicit solution.

Proposition 8. *If we assume that v_F satisfies Assumption 1, then, (i) v_F is a solution of Definition 11 $\iff$ (ii) v_F is an explicit solution of Definition 21.*

Proof of Proposition 8. Proof of $(i) \implies (ii)$. We assume that v_F is a solution of Definition 11. Given that Z^F satisfies Assumption 1, we can substitute the left-hand side of Equation (43) with Equation (11) to deduce

$$P_g^W\Big(\langle Z^F\rangle_t\Big) = F(t) \text{ for } t \geq 0. \tag{126}$$

Using Equation (124), Equation (126) can be reexpressed as

$$P_g^W\Big(v_F(t)\Big) = F(t) \text{ for } t \geq 0. \tag{127}$$

By Lemma 6, there exists an inverse $(P_g^W)^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P_g^W)^{-1}$ on both sides of Equation (127), Equation (127) can be rewritten as Equation (125).

Proof of $(ii) \implies (i)$. We assume that v_F is an explicit solution of Definition 21. We have

$$\begin{aligned} P_g^{Z^F}(t) &= P_g^W\Big(\langle Z^F\rangle_t\Big) \\ &= P_g^W(v_F(t)) \\ &= P_g^W((P_g^W)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}}) \\ &= F(t), \end{aligned}$$

where we use Equation (11) with the assumption that v_F satisfies Assumption 1 in the first equality, Equation (124) in the second equality, Equation (125) in the third equality, and algebraic manipulation in the fourth equality. $\square$

The following theorem states that under Assumption 5, (a) Z^F satisfies Assumption 1 and (b) that nondecreasing function is solution if and only if it is an explicit solution. The proof is a direct application of Proposition 7 and Proposition 8.

Theorem 14. *Under Assumption 5, (a) Z^F satisfies Assumption 1 (b) (i) v_F is a solution of Definition 11 $\iff$ (ii) v_F is an explicit solution of Definition 21.*

Proof of Theorem 14. To obtain (a), we apply Proposition 7 with Assumption 5. Then, an application of Proposition 8 with (a) yields (b). $\square$

Finally, we give the proof of Theorem 5, which is a direct consequence of Theorem 13.

Proof of Theorem 6. This is a direct consequence of Theorem 14 with Assumption 4. □

B.2. Two-sided nonrandom case

B.2.1. Case when the quadratic variation is absolutely continuous

The first lemma shows that there exists an inverse of $P^W_{g,h}$ which we denote $(P^W_{g,h})^{-1}$ and is strictly increasing such that $(P^W_{g,h})^{-1}(0) = 0$ and $(P^W_{g,h})^{-1}(1) = \infty$, all of which are new results which will be useful to prove the explicit solution of the inverse problem. The proof relies on Lemma 2.

Lemma 8. *There exists an inverse of $P^W_{g,h}$ which we denote $(P^W_{g,h})^{-1} : [0,1) \to \mathbb{R}^+$ and is strictly increasing such that $(P^W_{g,h})^{-1}(0) = 0$ and that $(P^W_{g,h})^{-1}(1) = \infty$.*

Proof of Lemma 8. Using Equation (17) from Lemma 2, we note that $P^W_{g,h} : \mathbb{R}^+ \to [0,1)$ is strictly increasing since $f^W_{g,h}(t) > 0$ for $t > 0$ by Equation (18). Thus, there exists an inverse$(P^W_{g,h})^{-1} : [0,1) \to \mathbb{R}^+$ which is strictly increasing. First, note that as $P^W_{g,h}(0) = 0$, this implies that $(P^W_{g,h})^{-1}(0) = 0$. Using Equation (17), some algebraic manipulation leads to $(P^W_{g,h})^{-1}(1) = \infty$. □

We then give another lemma whose proof relies on Lemma 2.

Lemma 9. *We have*

$$f^W_{g,h} \in \mathcal{C}_1(\mathbb{R}^+, \mathbb{R}^+) \ , \ P^W_{g,h} \in \mathcal{C}_1(\mathbb{R}^+, [0,1)) \text{ and} \tag{128}$$

$$(P^W_{g,h})^{-1} \in \mathcal{C}_1([0,1), \mathbb{R}^+). \tag{129}$$

Proof of Lemma 9. By Equations (17) and (18) in Lemma 2, we obtain Equations (128) and (129). □

We now give the definition of the explicit solution.

Definition 22. For a given pdf f, we say that a variance function $\sigma^2_f : \mathbb{R}^+ \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f \rangle_t = \int_0^t \sigma^2_{s,f} ds \text{ for } t \geq 0, \tag{130}$$

is an explicit solution if it is of the form Equation (48).

The next proposition shows that Assumption 6 implies that Z^f satisfies Assumption 1. The proof is mainly based on topological argument in $\mathbb{R}^+$ and the use of Assumption 6.

Proposition 9. *Under Assumption 6, we have that Z^f satisfies Assumption 1.*

Proof of Proposition 9. To prove that Z^f satisfies Assumption 1, we first show that $\sigma_f^2 \in L_{1,\text{loc}}(\mathbb{R}^+)$, i.e., we have to show by definition that $\forall\, K \subset \mathbb{R}^+$, K compact, we have

$$\int_K \sigma_{t,f}^2 \, \mathrm{d}t < +\infty. \tag{131}$$

There is no loss of generality assuming that K has a closed interval form $K = [K_0, K_1]$ where $0 \leq K_0 < K_1$, since if not we can break K into a finite number of nonoverlapping closed intervals by the Bolzano-Weierstrass theorem and prove Equation (109) for each interval. We first consider the case where

$$0 \leq K_F^0 < K_0 < K_1. \tag{132}$$

Given the form of the explicit solution (48), Equation (131) can be reexpressed as

$$\int_K \frac{f(t)}{f_{g,h}^W((P_{g,h}^W)^{-1}(F(t)))} \, \mathrm{d}t < +\infty. \tag{133}$$

We first show that the denominator in the integral of Equation (133) is uniformly bounded away from 0 on K. By Definition 9, F is a cdf and thus a nondecreasing function. We can deduce that

$$F(K_0) \leq F(t) \leq F(K_1) \text{ for } t \in K. \tag{134}$$

We also obtain by definition of K_F^0 in Equation (36), definition of K_F^1 in Equation (37) and the assumption that K_F^1 is not finite from Assumption 6 that

$$0 < F(\widetilde{K}) < 1 \text{ for } \widetilde{K} \in \mathbb{R} \text{ such that } K_F^0 < \widetilde{K}. \tag{135}$$

Combining Equations (134) and (135), we can deduce that

$$0 < F(K_0) \leq F(t) \leq F(K_1) < 1, \text{ for } t \in K. \tag{136}$$

By Lemma 8, we have that $(P_{g,h}^W)^{-1}$ is strictly increasing. Thus, applying $(P_{g,h}^W)^{-1}$ to each term of Inequality (136) yields

$$0 < (P_{g,h}^W)^{-1}(F(K_0)) \leq (P_{g,h}^W)^{-1}(F(t))) \leq (P_{g,h}^W)^{-1}(F(K_1)), \text{ for } t \in K. \tag{137}$$

We have that $(P_{g,h}^W)^{-1}(F(t)))$ takes its values in the closed interval

$$[(P_{g,h}^W)^{-1}(F(K_0)), (P_{g,h}^W)^{-1}(F(K_1))]$$

of $\mathbb{R}^+$ which is connected and compact by the Bolzano-Weierstrass theorem. Besides, it is known from topological properties that the image of a compact and connected set of $\mathbb{R}^+$ by a continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ is a compact and connected set of $\mathbb{R}^+$. Since $f_{g,h}^W$ is continuous by Equation (18), we can deduce that $f_{g,h}^W((P_{g,h}^W)^{-1}(F(t)))$ for $t \in K$ is included into a compact and connected space of $\mathbb{R}^+$, e.g., a closed interval of $\mathbb{R}^+$. From Equation (18), we get that there exists $C > 0$ such that

$$C \leq f^W_{g,h}((P^W_{g,h})^{-1}(F(t))) \text{ for } t \in K. \tag{138}$$

This implies that the denominator in the integral of Equation (133) is uniformly bounded away from 0 on K. Given that f is a pdf, we obtain that

$$\int_K f(t)\,\mathrm{d}t < +\infty.$$

Thus, Equation (133) holds. We now consider the general case when K is not necessarily of the form (132). We consider the case when $K_0 \leq K^0_F < K_1$. If we introduce the notation $\widetilde{K}^0_F = K^0_F + \eta^0_F$, then we can decompose $[K_0, K_1]$ as

$$[K_0, K_1] = [K_0, K^0_F] \cup [K^0_F, \widetilde{K}^0_F] \cup [\widetilde{K}^0_F, K_1].$$

We deduce that

$$\begin{aligned}
\int_K \sigma^2_{t,f}\,\mathrm{d}t &= \int_{[K_0,K^0_F]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[K^0_F,\widetilde{K}^0_F]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[\widetilde{K}^0_F,\widetilde{K}^1_F]} \sigma^2_{t,f}\,\mathrm{d}t \\
&= \int_{[K^0_F,\widetilde{K}^0_F]} \sigma^2_{t,f}\,\mathrm{d}t + \int_{[\widetilde{K}^0_F,\widetilde{K}^1_F]} \sigma^2_{t,f}\,\mathrm{d}t \\
&\leq C + \int_{[\widetilde{K}^0_F,\widetilde{K}^1_F]} \sigma^2_{t,f}\,\mathrm{d}t \\
&< +\infty,
\end{aligned}$$

where the second equality is due to the fact that the variance function is null on $[K_0, K^0_F]$ by Equation (48), the first inequality with $C > 0$ follows by Expression (47) from Assumption 6, and the second inequality is due to Equation (133). Finally, we have that the variance function is null on by Equation (48) in the case when $K^0_F \leq K_0$. We have thus shown Expression (131). Thus, we can deduce that Z^f is a local martingale with nonrandom quadratic variation

$$\langle Z^f \rangle_t = \int_0^t \sigma^2_{u,f} du \tag{139}$$

by Theorem I.4.40 (p. 48) from Jacod and Shiryaev (2003) with Expression (131). Finally, we show that $\langle Z^f \rangle_t \to \infty$ as $t \to \infty$. We can calculate that

$$\begin{aligned}
\langle Z^f \rangle_t &= \langle Z^F \rangle_t \\
&= v_F(t) \\
&= (P^W_{g,h})^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}}
\end{aligned} \tag{140}$$

where we use the fact that $Z^f = Z^F$ in the first equality, Equation (146) from Definition 23 in the second equality, and Definition 23 in the last equality. By Lemma 8 we have that $(P^W_{g,h})^{-1}(1) = \infty$, and by Definition 9 we have that

$\lim_{t\to\infty} F(t) = 1$. Thus, we can deduce by the assumption that K_F^1 is finite from Assumption 6 that

$$(P_{g,h}^W)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}} \to 0 \tag{141}$$

as $t \to \infty$. We can deduce by Equations (139), (140) and (141) that $\langle Z^f \rangle_t du \to \infty$ as $t \to \infty$. This implies that Z^f satisfies Assumption 1. $\square$

The next proposition states that if Z^f satisfies Assumption 1, then, the variance function is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (46) with Equations (19) from Theorem 2 and (130) and then differentiating and inverting on both sides of the equation to derive the explicit solution.

Proposition 10. *If we assume that Z^f satisfies Assumption 1, then, (i) σ_f^2 is a solution of Definition 12 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 22.*

Proof of Proposition 10. Proof of $(i) \implies (ii)$. We assume that σ_f^2 is a solution of Definition 12. Given that Z^f satisfies Assumption 1, we can substitute the left-hand side of Equation (46) with Equation (19) to deduce

$$P_{g,h}^W\Big(\langle Z^f \rangle_t\Big) = F(t) \text{ for } t \geq 0. \tag{142}$$

Using Equation (130), Equation (142) can be reexpressed as

$$P_{g,h}^W\Big(\int_0^t \sigma_{s,f}^2 ds\Big) = F(t) \text{ for } t \geq 0. \tag{143}$$

By Lemma 8, there exists an inverse$(P_{g,h}^W)^{-1} : [0, 1) \to \mathbb{R}^+$. Applying $(P_{g,h}^W)^{-1}$ on both sides of Equation (143), Equation (143) can be rewritten as

$$\int_0^t \sigma_{s,f}^2 ds = \quad (P_{g,h}^W)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}} \quad \text{ for } t \geq 0. \tag{144}$$

The left-hand side of Equation (144) and F have a derivative a.e. for $t \geq 0$ by absolute continuity properties and since F is absolutely continuous. $(P_{g,h}^W)^{-1}$ is differentiable on $[0, 1)$ by Lemma 9. Thus, we can differentiate Equation (144) a.e. on both sides, by using the chain rule on the right-hand side. We obtain

$$\sigma_{t,f}^2 = \quad f(t)((P_{g,h}^W)^{-1})'(F(t))\mathbf{1}_{\{0<F(t)<1\}} \quad \text{a.e. for } t \geq 0. \tag{145}$$

Applying the inverse function theorem, Equation (145) can be reexpressed as

$$\sigma_{t,f}^2 = \quad \frac{f(t)}{(P_{g,h}^W)'((P_{g,h}^W)^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{a.e. for } t \geq 0,$$

or equivalently of the form (48) as $(P_{g,h}^W)'(t) = f_{g,h}(t)$ a.e. for $t \geq 0$. Thus, we have shown that σ_f^2 is an explicit solution of Definition 22.

Proof of $(ii) \implies (i)$. We assume that σ_f^2 is an explicit solution of Definition 22. We have a.e. for $t \geq 0$

$$\begin{aligned} P_{g,h}^{Z^f}(t) &= P_{g,h}^W\Big(\int_0^t \sigma_{s,f}^2 ds\Big) \\ &= P_{g,h}^W\Big(\int_0^t \frac{f(s)}{f_{g,h}^W((P_{g,h}^W)^{-1}(F(s)))} \mathbf{1}_{\{0<F(t)<1\}} ds\Big) \\ &= P_{g,h}^W\Big(\int_0^t f(s)((P_{g,h}^W)^{-1})'(F(s)) \mathbf{1}_{\{0<F(t)<1\}} ds\Big) \\ &= P_{g,h}^W((P_{g,h}^W)^{-1})(F(t)) \\ &= F(t), \end{aligned}$$

where we use Equation (19) with the assumption that Z^f satisfies Assumption 1 in the first equality, Equation (48) in the second equality, the inverse function theorem in the third equality, integration in the fourth equality and algebraic manipulation in the fifth equality. We have thus shown that σ_t^2 satisfies Equation (45), and thus that σ_f^2 is a solution of Definition 12. □

The following theorem in the particular case when the quadratic variation $\langle Z \rangle$ is absolutely continuous states that under Assumption 6, (a) Z^f satisfies Assumption 1 and (b) that variance function is solution if and only if it is an explicit solution.

Theorem 15. *Under Assumption 6, (a) Z^f satisfies Assumption 1 (b) (i) σ_f^2 is a solution of Definition 12 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 22.*

Proof of Theorem 15. To obtain (a), we apply Proposition 9 with Assumption 6. Then, an application of Proposition 10 with (a) yields (b). □

Finally, we give the proof of Theorem 7, which is a direct consequence of Theorem 15.

Proof of Theorem 7. This is a direct consequence of Theorem 15 with Assumption 6. □

B.2.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of the explicit solution.

Definition 23. For a given cdf F, we say that a nonrandom nondecreasing function v_F which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F \rangle_t = v_F(t) \text{ for } t \geq 0, \tag{146}$$

is an explicit solution if it is of the form

$$v_F(t) = (P_{g,h}^W)^{-1}(F(t)) \mathbf{1}_{\{0<F(t)<1\}} \quad \text{for } t \geq 0. \tag{147}$$

The next proposition shows that Assumption 5 implies that Z^F satisfies Assumption 1.

Proposition 11. *Under Assumption 5, we have that Z^F satisfies Assumption 1.*

Proof of Proposition 11. By Definition 23, Z^F is defined as a continuous local martingale with quadratic variation $\langle Z^F \rangle_t = v_F(t)$ for $t \geq 0$, which can be expressed as

$$v_F(t) = \quad (P^W_{g,h})^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}} \quad \text{ for } t \geq 0.$$

By Lemma 8 we have that $(P^W_{g,h})^{-1}(1) = \infty$, and by Definition 9 we have that $\lim_{t\to\infty} F(t) = 1$. Thus, we can deduce by Assumption 5 that $\lim_{t\to\infty} v_F(t) = \infty$. This implies that $\langle Z^F \rangle_\infty = \infty$ and thus that Z^F satisfies Assumption 1. □

The next proposition states that if a nondecreasing function satisfies Assumption 1, then it is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (50) with Equations (19) and (146) and then inverting on both sides of the equation to derive the explicit solution.

Proposition 12. *We assume that v_F satisfies Assumption 1. Then, (i) v_F is a solution of Definition 13 $\iff$ (ii) v_F is an explicit solution of Definition 23.*

Proof of Proposition 12. Proof of $(i) \implies (ii)$. We assume that v_F is a solution of Definition 13. Given that Z^F satisfies Assumption 1, we can substitute the left-hand side of Equation (50) with Equation (19) to deduce

$$P^W_{g,h}\Big(\langle Z^F \rangle_t\Big) = F(t) \text{ for } t \geq 0. \tag{148}$$

Using Equation (146), Equation (148) can be reexpressed as

$$P^W_{g,h}\Big(v_F(t)\Big) = F(t) \text{ for } t \geq 0. \tag{149}$$

By Lemma 8, there exists an inverse$(P^W_{g,h})^{-1} : [0, 1) \to \mathbb{R}^+$. Applying $(P^W_{g,h})^{-1}$ on both sides of Equation (149), Equation (149) can be rewritten as Equation (147).

Proof of $(ii) \implies (i)$. We assume that v_F is an explicit solution of Definition 23. We have

$$\begin{aligned} P^{Z^F}_{g,h}(t) &= P^W_{g,h}\Big(\langle Z^F \rangle_t\Big) \\ &= P^W_{g,h}(v_F(t)) \\ &= P^W_{g,h}((P^W_g)^{-1}(F(t))\mathbf{1}_{\{0<F(t)<1\}}) \\ &= F(t), \end{aligned}$$

where we use Equation (19) with the assumption that v_F satisfies Assumption 1 in the first equality, Equation (146) in the second equality, Equation (147) in the third equality, and algebraic manipulation in the fourth equality. □

The following theorem states that under Assumption 5, (a) Z^F satisfies Assumption 1 and (b) that a nondecreasing function is solution if and only if it is an explicit solution.

Theorem 16. *We assume that Assumption 5 holds. Then, we have that (a) Z^F satisfies Assumption 1 and (b) (i) v_F is a solution of Definition 13 $\Longleftrightarrow$ (ii) v_F is an explicit solution of Definition 23.*

Proof of Theorem 16. To obtain (a), we apply Proposition 11 with Assumption 5. Then, an application of Proposition 12 with (a) yields (b). □

Finally, we give the proof of Theorem 8, which is a direct consequence of Theorem 16.

Proof of Theorem 8. This is a direct consequence of Theorem 16 with Assumption 5. □

B.3. One-sided random case

B.3.1. Case when the quadratic variation is absolutely continuous

We now give the definition of the explicit solution.

Definition 24. For a given random pdf f, we say that a variance process $\sigma^2_f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Y^f, i.e.

$$\langle Y^f \rangle_t(\omega) \quad = \quad \int_0^t \sigma^2_{s,f}(\omega) ds \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{150}$$

is an explicit solution if it is of the form

$$\sigma^2_{t,f}(\omega) = \frac{f(t,\omega)}{f_1^W((P_1^W)^{-1}(F(t,\omega)))} \mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{a.e. for } t \geq 0 \text{ and } \omega \in \Omega. \tag{151}$$

If we substitute $(P_1^W)^{-1}$ in Equation (151) with Equation (102) from Lemma 6, we can reexpress the explicit solution as Equation (56).

The next proposition shows that Assumption 7 implies that Y^f satisfies Assumption 2. The proof is mainly based on the use of Assumption 7.

Proposition 13. *Under Assumption 7, we have that Y^f satisfies Assumption 2.*

Proof of Proposition 13. We can deduce that Y^f is a local martingale with random quadratic variation

$$\langle Y^f \rangle_t(\omega) \quad = \quad \int_0^t \sigma^2_{u,f}(\omega) du \text{ for } t \geq 0 \text{ and } \omega \in \Omega \tag{152}$$

by Theorem I.4.40 (p. 48) from Jacod and Shiryaev (2003) with Expression (55) from Assumption 7. We show that $\langle Y^f\rangle_t \to \infty$ as $t \to \infty$. We can calculate that

$$\begin{aligned}\langle Y^f\rangle_t(\omega) &= \langle Y^F\rangle_t(\omega)\\ &= v_F(t,\omega)\\ &= \frac{1}{2\,\mathrm{erfinv}(1-F(t,\omega))^2}\mathbf{1}_{\{0<F(t,\omega)<1\}} \text{ for } t\geq 0 \text{ and } \omega\in\Omega,\end{aligned} \tag{153}$$

where we use the fact that $Y^f = Y^F$ in the first equality, Equation (159) from Definition 25 in the second equality, and Equation (61) in the last equality. By definition we have that $\mathrm{erfinv}(z) \to 0$ as $z \to 0$, and by Definition 14 we have that $\lim_{t\to\infty} F(t,\omega) = 1$. Thus, we can deduce by the assumption that K_F^1 is finite from Assumption 7 that

$$\frac{1}{2\,\mathrm{erfinv}(1-F(t,\omega))^2}\mathbf{1}_{\{0<F(t,\omega)<1\}} \to 0 \tag{154}$$

as $t \to \infty$. We can deduce by Equations (152), (153) and (154) that $\langle Y^f\rangle_t \to \infty$ as $t \to \infty$. This implies that Y^f satisfies Assumption 2. □

The next proposition states that if Y^f satisfies Assumption 2, then, the variance function is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (54) with Equations (26) from Theorem 3 and (150) and then differentiating and inverting on both sides of the equation to derive the explicit solution.

Proposition 14. *We assume that Y^f satisfies Assumption 2. Then, (i) σ_f^2 is a solution of Definition 16 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 24.*

Proof of Proposition 14. Proof of $(i) \implies (ii)$. We assume that σ_f^2 is a solution of Definition 16. Given that Y^f satisfies Assumption 2, we can substitute the left-hand side of Equation (54) with Equation (26) to deduce

$$P_1^W\Big(\langle Z^f\rangle_t(\omega)\Big) = F(t,\omega) \text{ for } t\geq 0 \text{ and } \omega\in\Omega. \tag{155}$$

Using Equation (150), Equation (155) can be reexpressed as

$$P_1^W\Big(\int_0^t \sigma_{s,f}^2(\omega)ds\Big) = F(t,\omega) \text{ for } t\geq 0 \text{ and } \omega\in\Omega. \tag{156}$$

By Lemma 6, there exists an inverse$(P_1^W)^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P_1^W)^{-1}$ on both sides of Equation (156), Equation (156) can be rewritten as

$$\int_0^t \sigma_{s,f}^2(\omega)ds = \quad (P_1^W)^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \text{for } t\geq 0 \text{ and } \omega\in\Omega. \tag{157}$$

The left-hand side of Equation (157) and F have a derivative a.e. for $t \geq 0$ by absolute continuity properties and since F is absolutely continuous. $(P_1^W)^{-1}$ is differentiable on $[0, 1)$ by Lemma 7. Thus, we can differentiate Equation (157) a.e. on both sides, by using the chain rule on the right-hand side. We obtain

$$\sigma^2_{t,f}(\omega) = f(t,\omega)((P_1^W)^{-1}(\omega))'(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{a.e. for } t \geq 0 \text{ and } \omega \in \Omega. \tag{158}$$

Applying the inverse function theorem, Equation (158) can be reexpressed as

$$\sigma^2_{t,f}(\omega) = \frac{f(t,\omega)}{(P_1^W)'((P_1^W)^{-1}(F(t,\omega)))}\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{a.e. for } t \geq 0 \text{ and } \omega \in \Omega,$$

or equivalently of the form (151) as $(P_1^W)'(t) = f_1(t)$ a.e. for $t \geq 0$. Thus, we have shown that σ^2_f is an explicit solution of Definition 24.

Proof of $(ii) \implies (i)$. We assume that σ^2_f is an explicit solution of Definition 24. We have a.e. for $t \geq 0$ and $\omega \in \Omega$ that

$$\begin{aligned}
P_1^{Y^f}(t|\omega) &= P_1^W(\int_0^t \sigma^2_{s,f}(\omega)ds) \\
&= P_1^W(\int_0^t \frac{f(s,\omega)}{f_1^W((P_1^W)^{-1}(F(s,\omega)))}\mathbf{1}_{\{0<F(s,\omega)<1\}}ds) \\
&= P_1^W(\int_0^t f(s,\omega)((P_1^W)^{-1})'(F(s,\omega))\mathbf{1}_{\{0<F(s,\omega)<1\}}ds) \\
&= P_1^W((P_1^W)^{-1})(F(t,\omega)) \\
&= F(t,\omega).
\end{aligned}$$

where we use Equation (26) with the assumption that Y^f satisfies Assumption 2 in the first equality, Equation (151) in the second equality, the inverse function theorem in the third equality, integration in the fourth equality and algebraic manipulation in the fifth equality. We have thus shown that σ^2_t satisfies Equation (53), and thus that σ^2_f is a solution of Definition 16. □

The following theorem states that under Assumption 7, (a) Y^f satisfies Assumption 2 and (b) that variance function is solution if and only if it is an explicit solution.

Theorem 17. *We assume that Assumption 7 holds. Then, we have that (a) Y^f satisfies Assumption 2 and (b) (i) σ^2_f is a solution of Definition 16 $\iff$ (ii) σ^2_f is an explicit solution of Definition 24.*

Proof of Theorem 17. To obtain (a), we apply Proposition 13 with Assumption 7. Then, an application of Proposition 14 with (a) yields (b). □

Finally, we give the proof of Theorem 9, which is a direct consequence of Theorem 17.

Proof of Theorem 9. This is a direct consequence of Theorem 17 with Assumption 7. □

B.3.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of the explicit solution.

Definition 25. For a given random cdf F, we say that a nondecreasing stochastic process v_F which is the quadratic variation of a continuous local martingale Y^F, i.e.

$$\langle Y^F\rangle_t(\omega) \quad = \quad v_F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{159}$$

is an explicit solution if it is of the form

$$v_F(t,\omega) = \quad (P_1^W)^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{160}$$

If we substitute $(P_1^W)^{-1}$ in Equation (160) with Equation (102) from Lemma 6, we can reexpress the explicit solution as Equation (61).

The next proposition shows that Assumption 8 implies that Y^F satisfies Assumption 2.

Proposition 15. *Under Assumption 8, we have that Y^F satisfies Assumption 2.*

Proof of Proposition 15. By Definition 25, Y^F is defined as a continuous local martingale with quadratic variation $\langle Y^F\rangle_t(\omega) = v_F(t,\omega)$ for $t \geq 0$ and $\omega \in \Omega$, which can be expressed by Equation (61) as

$$v_F(t,\omega) = \quad \tfrac{1}{2\operatorname{erfinv}(1-F(t,\omega))^2}\mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \text{for } t \geq 0 \text{ and } \omega \in \Omega.$$

By definition we have that $\operatorname{erfinv}(z) \to 0$ as $z \to 0$, and by Definition 14 we have that $\lim\limits_{t\to\infty} F(t,\omega) = 1$. Thus, we can deduce by Assumption 8 that $\lim\limits_{t\to\infty} v_F(t,\omega) = \infty$. This implies that $\langle Y^F\rangle_\infty = \infty$ and thus that Y^F satisfies Assumption 2. $\square$

The next proposition states that if a nondecreasing function satisfies Assumption 2, then it is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (58) with Equations (26) and (159) and then inverting on both sides of the equation to derive the explicit solution.

Proposition 16. *We assume that v_F satisfies Assumption 2. Then, we have that (i) v_F is a solution of Definition 17 $\iff$ (ii) v_F is an explicit solution of Definition 25.*

Proof of Proposition 16. Proof of $(i) \implies (ii)$. We assume that v_F is a solution of Definition 17. Given that Y^F satisfies Assumption 2, we can substitute the left-hand side of Equation (58) with Equation (26) to deduce

$$P_1^W\Big(\langle Y^F\rangle_t(\omega)\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{161}$$

Using Equation (159), Equation (161) can be reexpressed as

$$P_1^W\Big(v_F(t,\omega)\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{162}$$

By Lemma 6, there exists an inverse$(P_1^W)^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P_1^W)^{-1}$ on both sides of Equation (162), Equation (162) can be rewritten as Equation (160).

Proof of $(ii) \implies (i)$. We assume that v_F is an explicit solution of Definition 25. We have

$$\begin{aligned} P_1^{Y^F}(t|\omega) &= P_1^W\Big(\langle Y^F\rangle_t(\omega)\Big) \\ &= P_1^W\Big(v_F(t,\omega)\Big) \\ &= P_1^W\Big((P_1^W)^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}})\Big) \\ &= F(t,\omega), \end{aligned}$$

where we use Equation (26) with the assumption that v_F satisfies Assumption 2 in the first equality, Equation (159) in the second equality, Equation (160) in the third equality, and algebraic manipulation in the fourth equality. □

The following theorem states that under Assumption 8, (a) Y^F satisfies Assumption 2 and (b) that random nondecreasing function is solution if and only if it is an explicit solution.

Theorem 18. *We assume that Assumption 8 holds. Then, we have that (a) Y^F satisfies Assumption 2 and (b) (i) v_F is a solution of Definition 17 $\iff$ (ii) v_F is an explicit solution of Definition 25.*

Proof of Theorem 18. To obtain (a), we apply Proposition 15 with Assumption 8. Then, an application of Proposition 16 with (a) yields (b). □

Finally, we give the proof of Theorem 10, which is a direct consequence of Theorem 18.

Proof of Theorem 10. This is a direct consequence of Theorem 18 with Assumption 8. □

B.4. Two-sided random case

B.4.1. Case when the quadratic variation is absolutely continuous

We first give the definition of the explicit solution.

Definition 26. For a given random pdf f, we say that a variance process $\sigma_f^2 : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ which is the quadratic variation derivative a.e. of a continuous local martingale Z^f, i.e.

$$\langle Z^f\rangle_t(\omega) = \int_0^t \sigma_{s,f}^2(\omega)ds \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{163}$$

is an explicit solution if it is equal to Equation (64).

The next proposition shows that Assumption 7 implies that Z^f satisfies Assumption 2. The proof is mainly based on the use of Assumption 7.

Proposition 17. *Under Assumption 7, we have that Z^f satisfies Assumption 2.*

Proof of Proposition 17. We can deduce that Z^f is a local martingale with random quadratic variation

$$\langle Z^f \rangle_t(\omega) = \int_0^t \sigma_{u,f}^2(\omega) du \text{ for } t \geq 0 \text{ and } \omega \in \Omega \tag{164}$$

by Theorem I.4.40 (p. 48) from Jacod and Shiryaev (2003) with Expression (55) from Assumption 7. We show that $\langle Z^f \rangle_t \to \infty$ as $t \to \infty$. We can calculate that

$$\begin{aligned} \langle Z^f \rangle_t(\omega) &= \langle Z^F \rangle_t(\omega) \\ &= v_F(t, \omega) \\ &= (P_{g,h}^W)^{-1}(F(t,\omega)) \mathbf{1}_{\{0 < F(t,\omega) < 1\}} \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \end{aligned} \tag{165}$$

where we use the fact that $Z^f = Z^F$ in the first equality, Equation (159) from Definition 25 in the second equality, and Equation (172) in the last equality. By Lemma 8 we have that $(P_{g,h}^W)^{-1}(1) = \infty$, and by Definition 14 we have that $\lim_{t\to\infty} F(t,\omega) = 1$. Thus, we can deduce by the assumption that K_F^1 is finite from Assumption 7 that

$$(P_{g,h}^W)^{-1}(F(t,\omega)) \mathbf{1}_{\{0 < F(t,\omega) < 1\}} \to 0 \tag{166}$$

as $t \to \infty$. We can deduce by Equations (164), (165) and (166) that $\langle Z^f \rangle_t \to \infty$ as $t \to \infty$. This implies that Z^f satisfies Assumption 2. $\square$

The next proposition states that if Z^f satisfies Assumption 2, then, the variance function is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (63) with Equations (31) from Theorem 4 and (163) and then differentiating and inverting on both sides of the equation to derive the explicit solution.

Proposition 18. *We assume that Z^f satisfies Assumption 2. Then, we have that (i) σ_f^2 is a solution of Definition 18 $\iff$ (ii) σ_f^2 is an explicit solution of Definition 26.*

Proof of Proposition 18. Proof of $(i) \implies (ii)$. We assume that σ_f^2 is a solution of Definition 18. Given that Z^f satisfies Assumption 2, we can substitute the left-hand side of Equation (63) with Equation (31) to deduce

$$P_{g,h}^W\Big(\langle Z^f \rangle_t(\omega)\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{167}$$

Using Equation (163), Equation (167) can be reexpressed as

$$P_{g,h}^W\Big(\int_0^t \sigma_{s,f}^2(\omega) ds\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{168}$$

By Lemma 6, there exists an inverse $(P^W_{g,h})^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P^W_{g,h})^{-1}$ on both sides of Equation (168), Equation (168) can be rewritten as

$$\int_0^t \sigma^2_{s,f}(\omega)ds = (P^W_{g,h})^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{169}$$

The left-hand side of Equation (169) and F have a derivative a.e. for $t \geq 0$ by absolute continuity properties and since F is absolutely continuous. $(P^W_{g,h})^{-1}$ is differentiable on $[0,1)$ by Lemma 7. Thus, we can differentiate Equation (169) a.e. on both sides, by using the chain rule on the right-hand side. We obtain

$$\sigma^2_{t,f}(\omega) = f(t,\omega)((P^W_{g,h})^{-1}(\omega))'(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \tag{170}$$
$$\text{a.e. for } t \geq 0 \text{ and } \omega \in \Omega.$$

Applying the inverse function theorem, Equation (170) can be reexpressed as

$$\sigma^2_{t,f}(\omega) = \frac{f(t,\omega)}{(P^W_{g,h})'((P^W_{g,h})^{-1}(F(t,\omega)))}\mathbf{1}_{\{0<F(t,\omega)<1\}} \quad \text{a.e. for } t \geq 0 \text{ and } \omega \in \Omega,$$

or equivalently of the form (151) as $(P^W_{g,h})'(t) = f_{g,h}(t)$ a.e. for $t \geq 0$. Thus, we have shown that σ^2_f is an explicit solution of Definition 26.

Proof of $(ii) \implies (i)$. We assume that σ^2_f is an explicit solution of Definition 26. We have a.e. for $t \geq 0$ and $\omega \in \Omega$ that

$$\begin{aligned}
P^{Z^f}_{g,h}(t|\omega) &= P^W_{g,h}(\int_0^t \sigma^2_{s,f}(\omega)ds) \\
&= P^W_{g,h}(\int_0^t \frac{f(s,\omega)}{f^W_{g,h}((P^W_{g,h})^{-1}(F(s,\omega)))}\mathbf{1}_{\{0<F(s,\omega)<1\}}ds) \\
&= P^W_{g,h}(\int_0^t f(s)((P^W_{g,h})^{-1})'(F(s,\omega))\mathbf{1}_{\{0<F(s,\omega)<1\}}ds) \\
&= P^W_{g,h}((P^W_{g,h})^{-1})(F(t,\omega)) \\
&= F(t,\omega).
\end{aligned}$$

where we use Equation (31) with the assumption that Z^f satisfies Assumption 2 in the first equality, Equation (64) in the second equality, the inverse function theorem in the third equality, integration in the fourth equality and algebraic manipulation in the fifth equality. We have thus shown that σ^2_t satisfies Equation (62), and thus that σ^2_f is a solution of Definition 18. □

The following theorem states that under Assumption 7, (a) Z^f satisfies Assumption 2 and (b) that variance function is solution if and only if it is an explicit solution.

Theorem 19. *We assume that Assumption 7 holds. Then, we have (a) Z^f satisfies Assumption 2 and (b) (i) σ^2_f is a solution of Definition 18 $\iff$ (ii) σ^2_f is an explicit solution of Definition 26.*

Proof of Theorem 19. To obtain (a), we apply Proposition 17 with Assumption 7. Then, an application of Proposition 18 with (a) yields (b). □

Finally, we give the proof of Theorem 11, which is a direct consequence of Theorem 19.

Proof of Theorem 11. This is a direct consequence of Theorem 19 with Assumption 7. □

B.4.2. Case when the quadratic variation is not absolutely continuous

We first give the definition of the explicit solution.

Definition 27. For a given random cdf F, we say that a nondecreasing stochastic process v_F which is the quadratic variation of a continuous local martingale Z^F, i.e.

$$\langle Z^F\rangle_t(\omega) \quad = \quad v_F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega, \tag{171}$$

is an explicit solution if it is of the form

$$v_F(t,\omega) = \quad (P^W_{g,h})^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \text{for } t \geq 0 \text{ and } \omega \in \Omega. \tag{172}$$

The next proposition shows that Assumption 8 implies that Z^F satisfies Assumption 2.

Proposition 19. *Under Assumption 8, we have that Z^F satisfies Assumption 2.*

Proof of Proposition 19. By Definition 27, Z^F is defined as a continuous local martingale with quadratic variation $\langle Z^F\rangle_t(\omega) = v_F(t,\omega)$ for $t \geq 0$ and $\omega \in \Omega$, which can be expressed as

$$v_F(t,\omega) = \quad (P^W_{g,h})^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}} \qquad \text{for } t \geq 0.$$

By Lemma 8 we have that $(P^W_{g,h})^{-1}(1) = \infty$, and by Definition 14 we have that $\lim_{t\to\infty} F(t,\omega) = 1$. Thus, we can deduce by Assumption 8 that $\lim_{t\to\infty} v_F(t,\omega) = \infty$. This implies that $\langle Z^F\rangle_\infty = \infty$ and thus that Z^F satisfies Assumption 2. □

The next proposition states that if a nondecreasing function satisfies Assumption 2, then it is a solution if and only if it is an explicit solution. The proof is based on substituting the left-hand side of Equation (66) with Equations (31) and (171) and then inverting on both sides of the equation to derive the explicit solution.

Proposition 20. *We assume that v_F satisfies Assumption 2. Then, we have that (i) v_F is a solution of Definition 19 $\iff$ (ii) v_F is an explicit solution of Definition 27.*

Proof of Proposition 20. Proof of $(i) \implies (ii)$. We assume that v_F is a solution of Definition 19. Given that Z^F satisfies Assumption 2, we can substitute the left-hand side of Equation (66) with Equation (31) to deduce

$$P^W_{g,h}\Big(\langle Z^F\rangle_t(\omega)\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{173}$$

Using Equation (171), Equation (173) can be reexpressed as

$$P^W_{g,h}\Big(v_F(t,\omega)\Big) = F(t,\omega) \text{ for } t \geq 0 \text{ and } \omega \in \Omega. \tag{174}$$

By Lemma 6, there exists an inverse$(P^W_{g,h})^{-1} : [0,1) \to \mathbb{R}^+$. Applying $(P^W_{g,h})^{-1}$ on both sides of Equation (174), Equation (174) can be rewritten as Equation (172).

Proof of $(ii) \implies (i)$. We assume that v_F is an explicit solution of Definition 27. We have

$$\begin{aligned}
P^{Z^F}_{g,h}(t|\omega) &= P^W_{g,h}\Big(\langle Z^F\rangle_t(\omega)\Big) \\
&= P^W_{g,h}\Big(v_F(t,\omega)\Big) \\
&= P^W_{g,h}\Big((P^W_{g,h})^{-1}(F(t,\omega))\mathbf{1}_{\{0<F(t,\omega)<1\}}\Big) \\
&= F(t,\omega),
\end{aligned}$$

where we use Equation (31) with the assumption that v_F satisfies Assumption 2 in the first equality, Equation (171) in the second equality, Equation (172) in the third equality, and algebraic manipulation in the fourth equality. □

The following theorem states that under Assumption 8, (a) Z^F satisfies Assumption 2 and (b) that random nondecreasing function is solution if and only if it is an explicit solution.

Theorem 20. *We assume that Assumption 8 holds. Then, we have (a) Z^F satisfies Assumption 2 and (b) (i) v_F is a solution of Definition 19 $\iff$ (ii) v_F is an explicit solution of Definition 27.*

Proof of Theorem 20. To obtain (a), we apply Proposition 19 with Assumption 8. Then, an application of Proposition 20 with (a) yields (b). □