Approximation convergence in the inverse first-passage time problem

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Abstract: The inverse first-passage time problem determines a boundary such that the first-passage time of a Wiener process to this boundary has a given distribution. An approximation of the boundary by a piecewise linear boundary is given by equating the probability of the first-passage time to a linear boundary and the increment of the distribution on each interval. This is based on the boundary starting value, which is unknown in practice. We propose an approximation of the starting value of the boundary. We consider asymptotics where the length of each interval decreases. We first show that the approximation converges to the boundary starting value when assuming that the boundary is absolutely continuous and with positive starting value. We also show that a subsequence of the approximation uniformly converges to the boundary. A numerical study shows that the approximation is sensitive to the boundary starting value and slope, but is adequate.

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1. Introduction

This paper concerns the inverse first-passage time (IFPT) problem. The IFPT problem determines the boundary function such that the first-passage time (FPT) of a standard Wiener process to this boundary has a given distribution. This problem was formulated by A. Shiryaev during a Banach center meeting in 1976. More specifically, he considered the particular case of exponential distribution, which is commonly referred as the inverse Shiryaev problem.

The primary application of the IFPT problem is in portfolio credit risk modeling. Initially, the focus was on random walks (see Iscoe, Kreinin and Rosen (1999)). A detailed analysis of the IFPT problem and an approximation is given in Iscoe and Kreinin (1999). A model of default events with a randomized boundary is proposed in Schmidt and Novikov (2008). Another field of application is in pricing of credit derivatives (see Avellaneda and Zhu (2001) and Hull and White (2001)). The process represents the so-called distance to default of an obligor, while the FPT represents a default event. The boundary stands for a barrier separating the healthy states of the obligor from the default state. Another application is in inventory-control problem, whose formulation can be equivalent to the IFPT problem (see Song and Zipkin (2013)).

Despite their importance for applications, explicit solutions of the IFPT are very limited. They first exist when the boundary is linear. More specifically, Doob (1949) gives explicit formulae of crossing boundary probabilities (Equations (4.2)-(4.3), pp. 397-398) based on elementary geometrical and analytical arguments. They are obtained when the final time is not finite. Malmquist (1954) obtains an explicit formula conditioned on the starting and final values of the Wiener process for a finite final time (Theorem 1, p. 526). This is obtained with Doob's transformation (Section 5, pp. 401-402). Anderson (1960) derives an explicit formula conditioned on the final value of the Wiener process (Theorem 4.2, pp. 178-179). Then, he integrates it with respect to the final value of the Wiener process to get an explicit solution (Theorem 4.3, p. 180). For square root boundaries, Breiman (1967) expresses the FPT problem as an FPT of an Ornstein-Uhlenbeck process to a constant boundary. They are obtained with Doob's transformation. However, the boundary crossing probabilities of an Ornstein-Uhlenbeck process to a constant boundary are only known in the form of Laplace transform. Daniels (1969) uses the same technique and obtains an explicit solution. Potiron (2024+) obtains non-explicit formulae by the Girsanov theorem.

As explicit solutions are very limited, the literature related to the IFPT problem relies heavily on approximations (see Zucca and Sacerdote (2009) and Song and Zipkin (2011)). In Zucca and Sacerdote (2009), an approximation to a continuous boundary by a piecewise linear boundary is given by equating the probability of the FPT to a linear boundary and the increment of the cumulative distribution function (cdf) on each interval. That approximation uses Wang and Pötzelberger (1997) idea. That approximation is based on the starting value of the boundary, which has to be guessed in practice since it is unknown.

We propose an approximation of the boundary starting value, which makes it more suitable for applications. The idea is to equate the probability of the FPT to a constant boundary and the increment of the cdf on a first interval. First, we show that the approximation converges to the boundary starting value when assuming that the boundary is absolutely continuous and with positive starting value. Second, we show that a subsequence of the approximation uniformly converges to the boundary when assuming that the boundary is differentiable with uniformly bounded derivative. The results are obtained using Arzelà-Ascoli theorem on any compact space.

We consider asymptotics where the length of each interval of linear approximation goes to 0. These asymptotics are required to show that the approximation goes to the boundary asymptotically. The use of these asymptotics and the convergence results are new to the literature on the IFPT problem. They are important in practice, although we only obtain the convergence of a subsequence. A numerical study shows that the approximation is sensitive to the boundary starting value and slope, but is adequate. This also illustrates that these asymptotics are adapted to obtain an adequate approximation.

Since the formulation of the IFPT problem, many papers have investigated its theoretical properties. Dudley and Gutmann (1977) show the existence of a stopping time with respect to a general stochastic process, but this stopping time is not a FPT. The existence of lower semi-continuous solutions was established in Anulova (1981) for the FPT of a reflected Wiener process by compacity arguments in a discrete approximation of the boundary and the distribution. The IFPT problem is reformulated as a nonlinear Volterra integral equation in Peskir (2002a). Peskir (2002b) study the behavior in the neighborhood of 0. Abundo (2006) consider extensions to the general diffusion process case. When the distribution is non-atomic, Cheng et al. (2006) and Chen et al. (2011) show the existence and uniqueness of the IFPT problem for diffusions by a transfer into a free boundary problem. Jaimungal, Kreinin and Valov (2014) consider a connection between the Skorokhod embedding problem and the IFPT problem. For a general distribution, Ekström and Janson (2016) show the existence and uniqueness for Wiener processes by discretizing an optimal stopping problem. Beiglböck et al. (2018) consider a more general optimal stopping problem which yields existence and uniqueness as a by-product. Fukasawa and Obloj (2020) consider efficient discretisation of stochastic differential equations based on FPT of spheres. Chen, Chadam and Saunders (2022) study higher-order regularity properties of the solution of the IFPT problem. The uniqueness for reflected Wiener processes is shown by a discrete approximation argument along with stochastic ordering in Klump and Kolb (2023). The existence and the uniqueness for Levy processes and diffusions are studied in Klump and Savov (2023).

Our results also complement the theoretical results on continuous boundary in Chen et al. (2011) (Proposition 6) and Ekström and Janson (2016) (Theorem 8.2). Compared to these two papers, our approach based on compacity requires stronger assumptions. This is a price to pay as our approach is more direct and circumvents the use of a free boundary problem or optimal stopping theory. The results are also proved in the FPT problem of a reflected Wiener process, which are also new.

One limitation in this paper is that we only obtain the convergence of a subsequence. The reason is that we were not able to prove directly the convergence by extending the proving techniques used in Zucca and Sacerdote (2009). More specifically, their two main ideas are the use of concavity inequalities and the implicit function theorem. Assuming that the FPT cdf is absolutely continuous and the boundary is monotone concave, they prove in Theorem 4.3 (p. 1331) that the error due to the approximation is of the order equal to the maximum of the initial error and the squared interval length. We can weaken their assumptions on concavity by assumptions on differentiability with uniformly bounded derivatives. The elementary idea of the proof consists in bounding the difference between the approximation and the boundary value by a linear function on each interval. However, we were not able to extend their direct use of the implicit function theorem with the new asymptotics. The reason is that we need to use the implicit function theorem with an increasing number of approximation intervals, whereas they only use it with a finite number of approximation intervals. Although we were not able to track down the calculation, we conjecture that the direct convergence holds.

2. Setting

We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F}, \mathbf{F})$, where $\mathcal{F}$ is a σ-field and $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a filtration. For $A \subset \mathbb{R}^+$ and $B \subset \mathbb{R}$ such that $0 \in A$, we define the set of continuous functions with positive starting values as $\mathcal{C}_0^+(A, B) = \{h : A \to B \text{ s.t. } h \text{ is continuous and } h(0) > 0\}$. We first give the definition of the set of boundary functions. Since the approximation by a piecewise linear boundary given in Wang and Pötzelberger (1997) requires continuity of the boundary, we restrict ourselves to the continuous boundary case. Moreover, we do not allow for $h(0) = 0$ since our techniques, unfortunately, do not allow for that more complicated case.

Definition 1. We define the set of boundary functions as $\mathcal{G} = \mathcal{C}_0^+(\mathbb{R}^+, \mathbb{R})$.

We now give the definition of the FPT. We assume that the stochastic process is continuous since we consider a Wiener process or a reflected Wiener process in this paper.

Definition 2. We define the FPT of an $\mathbf{F}$-adapted continuous process $(Z_t)_{t \in \mathbb{R}^+}$ to a boundary $g \in \mathcal{G}$ as

$$\mathrm{T}_g^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g(t)\}. \tag{1}$$

We define an $\mathbf{F}$-standard Wiener process as $(W_t)_{t \in \mathbb{R}^+}$. We will consider the two cases in the following of this paper:

1. (Wiener process) $Z_t = W_t$
2. (reflected Wiener process) $Z_t = |W_t|$

We have that Z is a continuous and $\mathbf{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g(t)\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } (t, Z_t) \in G\}$, where $G = \{(t, u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u \geq g(t)\}$ is a closed subset of $\mathbb{R}^2$. Thus, the FPT T_g^Z is an $\mathbf{F}$-stopping time by Theorem I.1.27 (p. 7) in Jacod and Shiryaev (2003). We define the cdf of Z as

$$P_g^Z(t) = \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for any } t \geq 0. \tag{2}$$

The basic assumption for the approximation by a piecewise linear boundary given in Zucca and Sacerdote (2009) is that P_g^Z is absolutely continuous. Accordingly, the authors assume that all regularity assumptions ensuring the existence of the objects introduced and properties imposed are fulfilled. In the following assumption, we consider a slightly more explicit form.

Assumption 1. We assume that g is absolutely continuous on $\mathbb{R}^+$.

When g is continuous, we know by Theorem 8.1 in Ekström and Janson (2016) that P_g^Z is continuous. When g is continuously differentiable, we know by Lemma 3.3 in Strassen (1967) that P_g^Z is continuously differentiable. The following lemma shows that when g is absolutely continuous, then P_g^Z is absolutely continuous.

Lemma 1. *We assume that Assumption 1 holds. Then, P_g^Z is absolutely continuous on $\mathbb{R}^+$.*

Since P_g^Z is absolutely continuous, there exists a pdf $f_g^Z : \mathbb{R}^+ \to \mathbb{R}^+$, defined as

$$f_g^Z(t) = \frac{dP_g^Z(t)}{dt} \text{ for any } t \geq 0 \text{ a.e..} \tag{3}$$

We give the definition of possibly defective cdf. By Assumption 1, we naturally restrict ourselves to the absolute continuous cdf case.

Definition 3. A function $F : \mathbb{R}^+ \to [0,1]$ is a cdf if F is nondecreasing, absolutely continuous, i.e., with pdf $f : \mathbb{R}^+ \to \mathbb{R}^+$, defined as $f(t) = \frac{dF(t)}{dt}$ for any $t \geq 0$ a.e., and satisfies $F(0) = 0$ and $\lim_{t\to\infty} F(t) = F_\infty \leq 1$ where $0 < F_\infty \leq 1$.

The IFPT problem determines a boundary $g \in \mathcal{G}$ such that

$$f_g^Z(t) = f(t) \text{ for any } t \geq 0 \text{ a.e..} \tag{4}$$

As explicit solutions are very limited, the literature related to the IFPT problem relies heavily on approximations. Based on Wang and Pötzelberger (1997) idea, an approximation to a continuous boundary by a piecewise linear boundary is given in Zucca and Sacerdote (2009). Their driving idea is to determine recursively the slope of the linear approximation on an interval by equating the probability of the FPT of Z to the approximation and the increment of the cdf on the interval. That approximation is based on the starting value of the boundary. Since the starting value of the boundary is unknown, it has to be guessed in practice. Moreover, they do not propose any asymptotics when the length of each interval of linear approximation goes to 0. These asymptotics are required to show that the approximation goes to the boundary asymptotically.

In what follows, we consider a slight extension of their setting which approximates the starting value of the boundary, and with asymptotics where the length of each interval of linear approximation goes to 0. We define $t_f \in \mathbb{R}_*^+$ as the final time. For any $n \in \mathbb{N}$ and $m \in \{0, \ldots, 2^n\}$, we consider a time discretization $t_m^n = m\Delta_n$, where $\Delta_n = 2^{-n}t_f$ is the length of each interval of linear approximation. We consider a nested time discretization as this will be required in the proof of Theorem 4. We define the sequence of piecewise linear approximation of the boundary g^n recursively on m as

$$g^n(0) = \alpha_0^n, \tag{5}$$
$$g^n(u) = g^n(t_m^n) + \alpha_{m+1}^n(u - t_m^n) \text{ for any } u \in (t_m^n, t_{m+1}^n]. \tag{6}$$

Here, we have that $\alpha_0^n \in \mathbb{R}_*^+$ and $\alpha_m^n \in \mathbb{R}$ for $m \in \{1, \ldots, 2^n\}$ satisfy

$$P_{\alpha_0^n}^Z(\delta_n\Delta_n) = \int_0^{\delta_n\Delta_n} f(s)ds, \tag{7}$$
$$\mathbb{P}\big(T_{g^n}^Z \in [t_{m-1}^n, t_m^n]\big) = \int_{t_{m-1}^n}^{t_m^n} f(s)ds. \tag{8}$$

Equations (5)-(6) and Equation (8) correspond exactly to Equations (3.1)-(3.2) in Zucca and Sacerdote (2009). The novelty in this paper is Equation (7), in which we determine the approximation of the starting value of the boundary. The idea is to equate the probability of the FPT of Z to a constant boundary, equal to α_0^n, and the increment of the cdf on a first interval of proportion $0 < \delta_n < 1$. The reason why we introduce the tuning parameter δ_n is that there are numerical problems if we use $\delta_n = 1$. In practice, we recommend to use $\delta_n = 0.25$. There are no theoretical problems, since our main results do not require any assumption on the tuning parameter.

Our next result establishes that the sequence α_m^n is well-defined. This is a slight extension of Remark 3.2 in Zucca and Sacerdote (2009), which also includes that α_0^n is well-defined.

Proposition 2. *We assume that Assumption 1 holds. For any $n \in \mathbb{N}$, Equation (7) defines a unique $\alpha_0^n \in \mathbb{R}_*^+$ and Equation (8) defines a unique $\alpha_m^n \in \mathbb{R}$ for any $m \in \{0, \dots, 2^n\}$.*

3. Main results

We first show that the approximation converges to the boundary starting value. The elementary idea of the proof consists in observing that the boundary can be bounded below and above by positive constants for a very small time. Then, we show that these constants converge to the boundary starting value as the times goes to 0. This is possible since the boundary starting value is positive and the boundary is continuous, with our assumptions.

Proposition 3. *We assume that Assumption 1 holds. Then, the approximation converges to the boundary starting value, i.e., $g^n(0) \to g(0)$ as $n \to \infty$.*

We give our main result in the next theorem. This shows that a subsequence of the approximation uniformly converges to the boundary. The elementary idea of the proof consists in using Arzelà-Ascoli theorem on the compact space $[0, t_f]$. We first show that the α_m^n are uniformly bounded, which in turn implies that the approximated boundary is uniformly bounded and uniformly equicontinuous. For that purpose, we assume that the boundary is differentiable on $[0, t_f]$ with uniformly dominated derivative.

Assumption 2. We assume that g is differentiable on $[0, t_f]$ with uniformly bounded derivatives, i.e., $\sup\limits_{t \in [0, t_f]} \mid g'(t) \mid < \infty$.

One limitation in the next result is that we only obtain the convergence of a subsequence, rather than a direct convergence.

Theorem 4. *We assume that Assumption 2 holds. Then, there exists a subsequence g^{n_k} of g^n which converges uniformly to g on $[0, t_f]$, i.e., $\sup\limits_{t \in [0, t_f]} \mid g^{n_k}(t) - g(t) \mid \to 0$ as $n \to \infty$.*

4. Numerical study

In this section, we conduct a numerical study. We first show that the starting value approximation is sensitive to the boundary starting value and slope, but is adequate. Then, we check the stability of the approximation presented in Section 2 by means of some examples where a closed form solution is available. We also show another example where the solution is numerically evaluated.

First, we report in Table 1 the normalized error of the starting value approximation for several linear boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5$. The approximation is sensitive to the boundary starting value and slope. More specifically, the quality of approximation depends on the ratio of slope over value. When the slope is null, the absolute value of the normalized error is systematically below 0.10%. This can be explained by the fact that the approximated boundary is a constant. Overall, the normalized error is below 20.00% and with a positive bias for most instances. The case $\delta_n = 0.25$ reduces the normalized error by half compared to the case $\delta_n = 0.5$ for most instances. This illustrates that our asymptotics are adapted to obtain an adequate approximation of the starting value.

TABLE 1
Normalized error of the starting value approximation for several linear boundaries with $\Delta_n = 0.2$ *and* $\delta_n = 0.25, 0.5$

Boundary starting		$\delta_n = 0.25$				
Value	Slope	0	1	2	3	4
0.5		-0.10%	8.60%	17.38%	26.20%	35.04%
1		-0.10%	4.69%	9.49%	14.29%	19.10%
2		-0.10%	2.37%	4.84%	7.32%	9.79%
3		-0.10%	1.56%	3.22%	4.88%	6.54%
4		-0.10%	2.43%	4.92%	7.41%	9.89%
Boundary starting		$\delta_n = 0.5$				
Value	Slope	0	1	2	3	4
0.5		-0.06%	15.90%	32.14%	48.52%	64.92%
1		-0.05%	9.18%	18.47%	27.78%	37.10%
2		-0.05%	4.83%	9.73%	14.62%	19.52%
3		-0.05%	3.25%	6.55%	9.85%	13.15%
4		-0.05%	2.43%	4.92%	7.41%	9.89%

Second, we define the Daniels boundary and its pdf (see Daniels (1969)) for any $t \geq 0$ as

$$
\begin{aligned}
g(t) &= \frac{\alpha}{2} - \frac{t}{\alpha} \log\Big(\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \gamma \exp\big(-\frac{\alpha^2}{t}\big)}\Big), \\
f_g(t) &= \frac{1}{\sqrt{2\pi t^3}}\Big(\exp\big(-\frac{g(t)^2}{2t}\big) - \frac{2}{\beta}\exp\big(\frac{(g(t)-\alpha)^2}{2t}\big)\Big).
\end{aligned}
$$

Here, we have that $\alpha > 0$, $\beta \geq 0$ and $\gamma > \beta/4$. We also define the oscillating

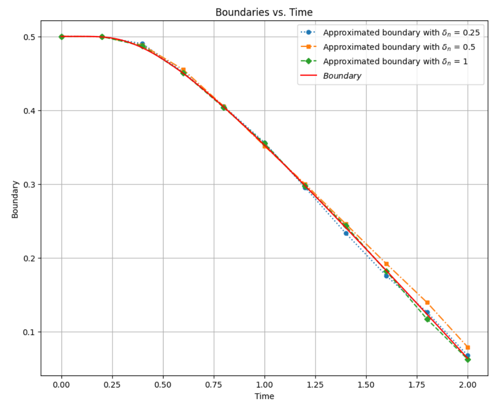

FIG 1. *Daniels boundary with parameters* $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$ *compared with approximated boundaries with* $\Delta_n = 0.2$ *and* $\delta_n = 0.25, 0.5, 1$

boundary for any $t \geq 0$ as

$$g(t) = \alpha + \beta \cos(\gamma t).$$

Since there is no explicit formula for f_g, we evaluate numerically its value by Buonocore, Nobile and Ricciardi (1987). Figure 1 plots the Daniels boundary with parameters $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$ compared with the approximated boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$. As the starting slope of the boundary is null, the three approximations are adequate. Figure 2 plots the Daniels boundary with parameters $\alpha = 1$, $\beta = 0.5$, $\gamma = 0.5$ compared with the approximated boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$. The starting slope is around unity. The approximation is adequate when $\delta_n = 0.25$, but not as accurate when $\delta_n = 0.5$, and a bit off when $\delta_n = 1$. As time increases, the inaccurate approximation tends to oscillate around the boundary. This is due to the fact that the approximation overcompensates by its own definition (8). This also illustrates that the asymptotics are adapted to obtain an adequate approximation. Figure 3 plots the oscillating boundary with parameters $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 8$ compared with approximated boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$. This is a more complicated case as the boundary is not monotone. As for Figure 2, The approximation is adequate when $\delta_n = 0.25$, but not as accurate when $\delta_n = 0.5$, and a bit off when $\delta_n = 1$. Moreover, all the approximations are a bit off at points where the monotonicity changes. This documents the limitation of the method of approximation. These results are confirmed by Table 2, which reports the mean squared error of the approximations for several boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$.

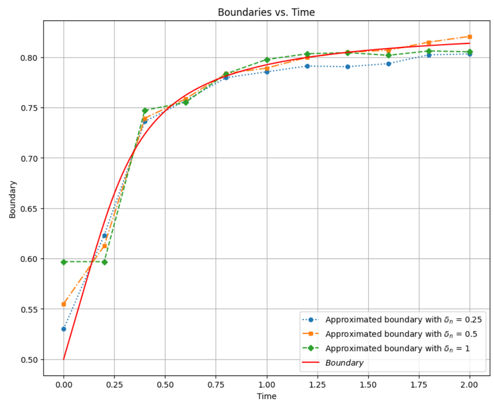

Fig 2. *Daniels boundary with parameters $\alpha = 1$, $\beta = 0.5$, $\gamma = 0.5$ compared with approximated boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$*

5. Proofs from the setting

In this section, we give the proofs from the setting, which are elementary. We start with the proof of Lemma 1, which slightly extends the arguments from the proof of Lemma 3.3 in Strassen (1967).

Proof of Lemma 1. By Assumption 1, we have that g is absolutely continuous on $\mathbb{R}^+$. Thus, g admits a derivative a.e. on $\mathbb{R}^+$, i.e., there exists a Lebesgue-negligible set $\mathcal{N} \subset \mathbb{R}^+$ such that g admits a derivative for any $t \in \mathbb{R}^+ - \mathcal{N}$. To show that P_g^W is absolutely continuous on $\mathbb{R}^+$, it is sufficient to show that P_g^W admits a derivative for any $t \in \mathbb{R}^+ - \mathcal{N}$, since $\mathcal{N}$ is a Lebesgue-negligible set. By the definition of absolute continuity, we have that $\mathcal{N}$ is countable on any compact space of $\mathbb{R}^+$. Indeed, if $s \in \mathbb{R}^+$ is an accumulation point of $\mathcal{N}$, then g does not admit a derivative in the neighborhood of s and thus g is not absolutely continuous. Thus, we have that $\mathcal{N}$ is countable on any compact space of $\mathbb{R}^+$. To show that P_g^W is absolutely continuous on $\mathbb{R}^+$, it is then sufficient to show that P_g^W admits a derivative on any open interval (u, v) where $u \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ satisfy $u < v$ and $(u, v) \cap \mathcal{N} = \emptyset$. We can show this statement by extending the arguments from the proof of Lemma 3.3 in Strassen (1967) along with the assumption that $g(0) > 0$ by Definition 1. The reflected Wiener process case follows since the FPT of a reflected Wiener process to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal.

□

In the next definition, we introduce the transition pdf of a stochastic process constrained by an absorbing boundary.

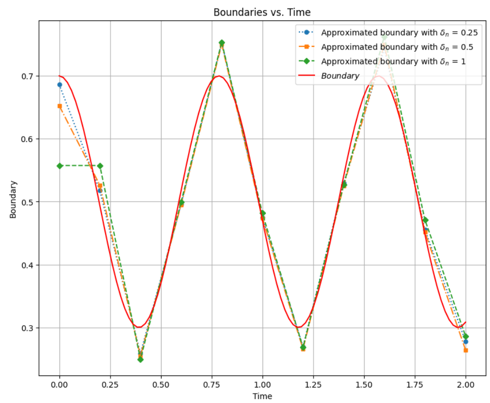

Fig 3. *Oscillating boundary with parameters $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 8$ compared with approximated boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$*

Definition 4. We define the transition pdf of the stochastic process Z at time t constrained by the absorbing boundary g over $[s,t]$ given that $Z_s = y$ as $p_g^Z(t, x \mid s, y)$ such that

$$p_g^Z(t, x \mid s, y) = \frac{\partial}{\partial x}\mathbb{P}(Z_t < x, \mathrm{T}_g^Z > t \mid Z_s = y) \tag{9}$$

with $x < g(t)$, $t > s \geq 0$ and $y < g(s)$ given and fixed.

In the following lemma, we give the pdf and the transition pdf for the FPT of a Wiener process to a linear boundary. This is a consequence to Doob (1949) (Equation (4.2), p. 397), Malmquist (1954) (p. 526) and Durbin (1971) (Lemma 1).

Lemma 5. *We assume that the boundary is linear*

$$g(t) = \alpha_1 t + \alpha_0 \text{ for any } t \geq 0.$$

Here, $t_0 \geq 0$, $\alpha_0 \in \mathbb{R}_*^+$, $\alpha_1 \in \mathbb{R}$, and $x_0 \in \mathbb{R}$ are such that $g(t_0) > x_0$. We have that the pdf for the FPT of a Wiener process is equal to

$$f_g^W(t \mid t_0, x_0) = \frac{\alpha_0 - x_0}{\sqrt{2\pi(t-t_0)^3}} \exp\Big(-\frac{(\alpha_0 + \alpha_1(t-t_0) - x_0)^2}{2(t-t_0)}\Big). \tag{10}$$

The transition pdf is equal to

$$p_g^W(t_1, x_1 \mid t_0, x_0) = \Big(1 - \exp\Big(\frac{-2(g(t_1)-x_1)(g(t_0)-x_0)}{t_1 - t_0}\Big)\Big) \frac{\exp(-\frac{(x_1-x_0)^2}{2(t_1-t_0)})}{\sqrt{2\pi(t_1-t_0)}}. \tag{11}$$

TABLE 2
Mean squared error of the approximations for several boundaries with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$

Daniels boundary with $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$	
δ_n	MSE
0.25	1.66×10^{-5}
0.5	5.96×10^{-5}
1	5.89×10^{-5}
Daniels boundary with $\alpha = 1$, $\beta = 0.5$, $\gamma = 0.5$	
δ_n	MSE
0.25	1.82×10^{-4}
0.5	3.56×10^{-4}
1	1.07×10^{-3}
Oscillating boundary with $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 8$	
δ_n	MSE
0.25	7.09×10^{-2}
0.5	6.96×10^{-2}
1	7.21×10^{-2}

Proof of Lemma 5. Equation (10) is obtained in Doob (1949) (Equation (4.2), p. 397) or Malmquist (1954) (p. 526). Equation (11) follows from Durbin (1971) (Lemma 1). □

In the following lemma, we give the pdf and the transition pdf for the FPT of a Wiener process to a continuous piecewise linear boundary. Equation (13) is already available in Wang and Pötzelberger (1997) and Zucca and Sacerdote (2009) (Section 2.1.3, pp. 1323-1324)

Lemma 6. *We assume that the boundary is piecewise linear*

$$g(t) = \alpha_i t + \beta_i, \text{ for any } t \in [t_{i-1}, t_i].$$

Here, we have that $t_i = i\Delta + t_0$, where $t_0 \geq 0$, $\Delta > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$ satisfy $\alpha_{i+1} + \beta_{i+1} t_i = \alpha_i + \beta_i t_i$, so that the boundary is continuous. We can express the transition pdf as

$$p_g^W(t_1, x_1, \ldots, t_n, x_n \mid t_0, x_0) = \prod_{i=1}^{n} p_g^W(t_i, x_i \mid t_{i-1}, x_{i-1}). \tag{12}$$

Here, we have that $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $x_i \leq g(t_i)$ for $i = 1, \ldots, n$ and $x_0 < g(t_0)$ where $t_0 < t_1 < t_2 < \ldots < t_n$ are given and fixed. We can reexpress the transition pdf with the following explicit expression

$$\begin{aligned} p_g^W(t_1, x_1, \ldots, t_n, x_n \mid t_0, x_0) &= \prod_{i=1}^{n} \Big(1 - \exp\big(\frac{-2(g(t_i) - x_i)(g(t_{i-1}) - x_{i-1})}{t_i - t_{i-1}}\big)\Big) \\ &\quad \times \frac{\exp\Big(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\Big)}{\sqrt{2\pi(t_i - t_{i-1})}}. \end{aligned} \tag{13}$$

We can deduce that

$$\mathbb{P}(W_{t_1} \in C_1, \ldots, W_{t_n} \in C_n, \mathrm{T}_g^W > t_n \mid W_{t_0} = x_0) \tag{14}$$
$$= \int_{C_1} \cdots \int_{C_n} p_g^W(t_1, x_1, \ldots, t_n, x_n \mid t_0, x_0) dx_1 \ldots dx_n$$

for any Borel set $C_i \subset (-\infty, g(t_i)]$ *with* $i = 1, \ldots, n$.

Proof of Lemma 6. Equation (12) is obtained by Definition (9) and follows by induction with conditional probability formula. Then, Equation (13) can be deduced by plugging Equation (11) into Equation (12). Finally, Equation (14) is a direct consequence of Equation (13). □

We define ϕ as the standard Gaussian cdf. In the following lemma, we give the pdf for the FPT of a reflected Wiener process to a linear boundary. This is based on the explicit solution from Anderson (1960) (Theorem 5.1, p. 191) for the FPT to an upper linear boundary and a lower linear boundary. This is due to the fact that the FPT of a reflected Wiener process to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal. Although we can deduce the transition pdf and transition pdf for the piecewise linear boundary with the same arguments as in the proofs of Lemma 5 and Lemma 6, we do not report them in the following of this paper.

Lemma 7. *We assume that the boundary is linear*

$$g(t) = \alpha_1 t + \alpha_0 \text{ for any } t \geq 0.$$

Here, we have that $t_0 \geq 0$, $\alpha_0 \in \mathbb{R}_*^+$, $\alpha_1 \in \mathbb{R}$, *and* $x_0 \in \mathbb{R}$ *are such that* $g(t_0) > x_0$. *We have that the pdf for the FPT of a reflected Wiener process is equal to*

$$f_g^{|W|}(t_0 \mid t_0, x_0) = 0, \tag{15}$$

$$\begin{aligned} f_g^{|W|}(t \mid t_0, x_0) = {} & \frac{2}{(t-t_0)^{3/2}} \phi\Big(\frac{\alpha_1(t-t_0) + \alpha_0 - x_0}{\sqrt{t-t_0}}\Big) \sum_{r=0}^{\infty} \Big\{ (4r+1)(\alpha_0 - x_0) \\ & \times \exp\Big(\frac{-(8r(r+1)(\alpha_0 - x_0))(\alpha_1(t-t_0) + \alpha_0 - x_0)}{t-t_0}\Big) \\ & -(4r+2)(\alpha_0 - x_0) \\ & \times \exp\Big(\frac{-(4(r+1)(2r+1)(\alpha_0 - x_0)(\alpha_1(t-t_0) + \alpha_0 - x_0)}{t-t_0}\Big)\Big\} \\ & \text{for any } t > t_0. \end{aligned} \tag{16}$$

Proof of Lemma 7. We first consider the FPT of a Wiener process to an upper linear boundary and a lower linear boundary. We first assume that the boundary is upper linear and lower linear, i.e., that $g(t) = (\gamma_2 + \delta_2 t, \gamma_1 + \delta_1 t)$, where

$\gamma_1 > 0$, $\gamma_2 < 0$, $\delta_1 \geq \delta_2$ and we do not have that $\delta_1 = \delta_2 = 0$. By Anderson (1960) (Theorem 5.1, p. 191), we have that the pdf of the FPT is equal to

$$
\begin{aligned}
f_g^W(t_0) &= 0, && (17)\\
f_g^W(t) &= \frac{1}{(t-t_0)^{3/2}}\Bigg[\phi\Big(\frac{\delta_1(t-t_0)+\gamma_1}{\sqrt{t-t_0}}\Big)\sum_{r=0}^{\infty}\Big\{\big((2r+1)\gamma_1 - 2r\gamma_2\big) && (18)\\
&\quad \times \exp\Big(\frac{-2r(r\gamma_1-(r+1)\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\\
&\quad -\big(2(r+1)\gamma_1 - 2r\gamma_2\big)\\
&\quad \exp\Big(\frac{-2(r+1)((r+1)\gamma_1-r\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\Big\}\\
&\quad +\phi\Big(\frac{\delta_2(t-t_0)+\gamma_2}{\sqrt{t-t_0}}\Big)\sum_{r=0}^{\infty}\Big\{\big(2r\gamma_1-(2r+1)\gamma_2\big)\\
&\quad \times \exp\Big(\frac{-2(r+1)((r+1)\gamma_1-r\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\\
&\quad -\big(2(r+1)\gamma_1 - 2r\gamma_2\big)\\
&\quad \exp\Big(\frac{-2r(r\gamma_1-(r+1)\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2 t+\gamma_2))}{t-t_0}\Big)\Big\}\Bigg],
\end{aligned}
$$

for any $t > t_0$. Now, we assume that the boundaries are symmetric, i.e., that $g(t) = (-\alpha_1 t - \alpha_0, \alpha_1 t + \alpha_0)$ where $\alpha_1 \in \mathbb{R}$ and $\alpha_0 \in \mathbb{R}_*^+$. From Equations (17)-(18), we can deduce that

$$
\begin{aligned}
f_g^W(t_0) &= 0, && (19)\\
f_g^W(t) &= \frac{2}{(t-t_0)^{3/2}}\phi\Big(\frac{\alpha_1(t-t_0)+\alpha_0}{\sqrt{t-t_0}}\Big) && (20)\\
&\quad \sum_{r=0}^{\infty}\Big\{(4r+1)\alpha_0\exp\Big(\frac{-(8r(r+1)\alpha_0)(\alpha_1 t - t_0 + \alpha_0)}{t-t_0}\Big)\\
&\quad -(4r+2)\alpha_0\exp\Big(\frac{-(4(r+1)(2r+1)\alpha_0(\alpha_1(t-t_0)+\alpha_0)}{t-t_0}\Big\} \text{ for any } t > t_0.
\end{aligned}
$$

We have the FPT of a reflected Wiener to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal. From the previous sentence and Equations (19)-(20), we can deduce Equations (15)-(16).

□

The next lemma gives the transition pdf for a FPT of a Wiener process W at time t_{m+1}^n constrained by the absorbing boundary g^n over $[t_m^n, t_{m+1}^n]$ given that $W_{t_m} = x_m$.

Lemma 8. *For any $n \in \mathbb{N}_*$ and $m \in \{0, \dots, 2^n - 1\}$, we have*

$$p^W_{g^n}\big(t^n_{m+1}, x_{m+1} \mid t^n_m, x_m\big) = \Big(1 - \exp\Big(\frac{-2(g^n(t^n_{m+1}) - x_{m+1})(g^n(t^n_m) - x_m)}{\Delta_n}\Big)\Big) \times \frac{\exp\big(-\frac{(x_{m+1}-x_m)^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}}. \quad (21)$$

Proof of Lemma 8. Equation (21) can be obtained directly from Equation (11) in Lemma 5. □

We define the probability of the FPT to a constant boundary equal to $\alpha \in \mathbb{R}^+_*$ on $[0, \delta_n \Delta_n]$ as $G^n_0 : \mathbb{R}^+_* \to \mathbb{R}$ such that

$$G^n_0(\alpha) = 1 - \int_{-\infty}^{\alpha} \Big(1 - \exp\Big(\frac{-2(\alpha - x_1)\alpha}{\delta_n \Delta_n}\Big)\Big) \frac{\exp\big(-\frac{x_1^2}{\delta_n \Delta_n}\big)}{\sqrt{\pi \delta_n \Delta_n}} dx_1. \quad (22)$$

The next lemma gives a more explicit form to α^n_0.

Lemma 9. *For any $n \in \mathbb{N}$, Equation (7) can be reexpressed as*

$$G^n_0(\alpha^n_0) - \int_0^{\delta_n \Delta_n} f(s) ds = 0. \quad (23)$$

Proof of Lemma 9. We have that

$$\begin{aligned} P^W_{\alpha^n_0}(\delta_n \Delta_n) &= \mathbb{P}\big(T^W_{\alpha^n_0} \in [0, \delta_n \Delta_n]\big) \\ &= \mathbb{P}\big((T^W_{\alpha^n_0} > \delta_n \Delta_n)^C\big) \\ &= 1 - \mathbb{P}\big(T^W_{\alpha^n_0} > \delta_n \Delta_n\big) \\ &= 1 - \mathbb{P}\big(W_{\delta_n \Delta_n} \in (-\infty, \alpha^n_0], T^W_{\alpha^n_0} > \delta_n \Delta_n\big) \\ &= 1 - \int_{-\infty}^{\alpha^n_0} p^W_{\alpha^n_0}(\delta_n \Delta_n, x_1 \mid 0, 0) dx_1 \\ &= 1 - \int_{-\infty}^{\alpha^n_0} \Big(1 - \exp\Big(\frac{-2(\alpha^n_0 - x_1)\alpha^n_0}{\delta_n \Delta_n}\Big)\Big) \frac{\exp\big(-\frac{x_1^2}{\delta_n \Delta_n}\big)}{\sqrt{\pi \delta_n \Delta_n}} dx_1 \end{aligned} \quad (24)$$

Here, we use Equation (2) in the first equality, the fact that $T^W_{\alpha^n_0} \geq 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the second equality, elementary probability facts in the third equality, the fact that $T^W_{\alpha^n_0} \subset \{W_{\delta_n \Delta_n} \in (-\infty, \alpha^n_0]\}$ in the fourth equality, Equation (14) from Lemma 6 in the fifth equality and Equation (21) from Lemma 8 in the sixth equality. Finally, we can deduce Equation (23) by plugging Equation (7) into Equation (24). □

We define the probability of the FPT to a linear boundary started from α^n_0 with trend $\alpha \in \mathbb{R}$ on $[0, t^n_1]$ as $G^n_1 : \mathbb{R} \to \mathbb{R}$ such that

$$G^n_1(\alpha) = 1 - \int_{-\infty}^{\alpha^n_0 + \alpha \Delta_n} \Big(1 - \exp\Big(\frac{-2(\alpha^n_0 + \alpha \Delta_n - x_1)\alpha^n_0}{\Delta_n}\Big)\Big) \frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} dx_1. \quad (25)$$

In the next lemma, we give a more explicit form to α_1^n based on the known value α_0^n.

Lemma 10. *For any $n \in \mathbb{N}$ and $m = 1$, Equation (8) can be reexpressed as*

$$G_1^n(\alpha_1^n) - \int_0^{t_1^n} f(s)ds = 0. \tag{26}$$

Proof of Lemma 10. We have that

$$\begin{aligned}
\mathbb{P}\big(T_{g^n}^W \in [0, t_1^n]\big) &= \mathbb{P}\big((T_{g^n}^W > t_1^n)^C\big) \\
&= 1 - \mathbb{P}\big(T_{g^n}^W > t_1^n\big) \\
&= 1 - \mathbb{P}\big(W_{t_1^n} \in (-\infty, g^n(t_1^n)], T_{g^n}^W > t_1^n\big) \\
&= 1 - \int_{-\infty}^{g^n(t_1^n)} p_{g^n}^W(t_1^n, x_1 \mid 0, 0) dx_1 \\
&= 1 - \int_{-\infty}^{g^n(t_1^n)} \Big(1 - \exp\Big(\frac{-2(g^n(t_1^n) - x_1) g^n(0)}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp\big(-\frac{x_1^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}} dx_1 \\
&= 1 - \int_{-\infty}^{\alpha_0^n + \alpha_1^n \Delta_n} \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \alpha_1^n \Delta_n - x_1)\alpha_0^n}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp\big(-\frac{x_1^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}} dx_1.
\end{aligned} \tag{27}$$

Here, we use the fact that $T_{g^n}^W \geq 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the first equality, elementary probability facts in the second equality, the fact that $T_{g^n}^W \subset \{W_{t_1^n} \in (-\infty, g^n(t_1^n)]\}$ in the third equality, Equation (14) from Lemma 6 in the fourth equality, Equation (21) from Lemma 8 in the fifth equality and Equations (5)-(6) in the sixth equality. Finally, we can deduce Equation (26) by plugging Equation (8) into Equation (27). □

We define the probability of the FPT to a continuous piecewise linear boundary g^n on $[0, t_1^n]$ and with trend $\alpha \in \mathbb{R}$ on $[t_1^n, t_2^n]$ as $G_2^n : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}
G_2^n(\alpha) &= 1 - G_1^n(\alpha_1^n) - \int_{-\infty}^{\alpha_0^n + \alpha_1^n \Delta_n} \int_{-\infty}^{\alpha_0^n + (\alpha_1^n + \alpha)\Delta_n} \\
&\quad \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + (\alpha_1^n + \alpha)\Delta_n - x_2)(\alpha_0^n + \alpha_1^n \Delta_n - x_1)}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp(-\frac{(x_2 - x_1)^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} \\
&\quad \times \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \alpha_1^n \Delta_n - x_1)\alpha_0^n}{\Delta_n}\Big)\Big) \frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} dx_1 dx_2.
\end{aligned} \tag{28}$$

In the next lemma, we give a more explicit form to α_2^n based on the known values α_1^n and α_0^n.

Lemma 11. *For any $n \in \mathbb{N}$ and $m = 2$, Equation (8) can be reexpressed as*

$$G_2^n(\alpha_2^n) - \int_{t_1^n}^{t_2^n} f(s)ds = 0.$$

Proof of Lemma 11. We have that

$$\begin{aligned}
\mathbb{P}\big(T_{g^n}^W \in [t_1^n, t_2^n]\big) &= \mathbb{P}\left((\mathrm{T}_{g^n}^W < t_1^n, \mathrm{T}_{g^n}^W > t_2^n)^C\right) \\
&= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_1^n, \mathrm{T}_{g^n}^W > t_2^n\right) \\
&= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_1^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\
&= 1 - \mathbb{P}\left(0 \le \mathrm{T}_{g^n}^W < t_1^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\
&= 1 - \int_0^{t_1^n} f(s)ds - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\
&= 1 - G_1^n(\alpha_1^n) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right). \qquad (29)
\end{aligned}$$

Here, we use elementary probability facts in the first and second equalities, the fact that $\{\mathrm{T}_{g^n}^W < \Delta_n\}$ and $\{\mathrm{T}_{g^n}^W > 2\Delta_n\}$ are disjoint events in the third equality, the fact that $T_{g^n}^W \ge 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the fourth equality, Equation (8) in the fifth equality, and Lemma 10 in the sixth equality. Also, we have that

$$\begin{aligned}
\mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) &= \mathbb{P}\big(W_{t_1^n} \in (-\infty, g^n(t_1^n)], W_{t_2^n} \in (-\infty, g^n(t_2^n)], \mathrm{T}_{g^n}^W > t_2^n\big) \\
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} p_{g^n}^W(t_1^n, x_1, t_2^n, x_2 \mid 0, 0) dx_1 dx_2 \\
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} p_{g^n}^W(t_2^n, x_2 \mid t_1^n, x_1) p_{g^n}^W(t_1^n, x_1 \mid 0, 0) dx_1 dx_2 \\
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} \\
&\quad \left(1 - \exp\left(\frac{-2(g^n(t_2^n) - x_2)(g^n(t_1^n) - x_1)}{\Delta_n}\right)\right) \frac{\exp\left(-\frac{(x_2 - x_1)^2}{\Delta_n}\right)}{\sqrt{\pi \Delta_n}}
\end{aligned}$$

$$\times(1-\exp(\frac{-2(g^n(t_1^n)-x_1)g^n(t_0^n)}{\Delta_n}))\frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi\Delta_n}}dx_1dx_2$$

$$= \int_{-\infty}^{\alpha_0^n+\alpha_1^n\Delta_n}\int_{-\infty}^{\alpha_0^n+(\alpha_1^n+\alpha_2^n)\Delta_n} \tag{30}$$

$$\Big(1-\exp\Big(\frac{-2(\alpha_0^n+(\alpha_1^n+\alpha_2^n)\Delta_n-x_2)(\alpha_0^n+\alpha_1^n\Delta_n-x_1)}{\Delta_n}\Big)\Big)$$

$$\frac{\exp(-\frac{(x_2-x_1)^2}{\Delta_n})}{\sqrt{\pi\Delta_n}}$$

$$\times\Big(1-\exp\Big(\frac{-2(\alpha_0^n+\alpha_1^n\Delta_n-x_1)\alpha_0^n}{\Delta_n}\Big)\Big)\frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi\Delta_n}}dx_1dx_2.$$

Here, we use the fact that

$$\{\mathrm{T}_{g^n}^W>t_2^n\}\subset\{W_{t_1^n}\in(-\infty,g^n(t_1^n)],W_{t_2^n}\in(-\infty,g^n(t_2^n)]\}$$

in the first equality, Equation (14) in the second equality, Equation (12) in the third equality, Equation (21) from Lemma 8 in the fourth equality and Equations (5)-(6) in the fifth equality. Finally, we can deduce Equation (29) by plugging Equation (30) and Equation (8) into Equation (29). □

For any $m\in\{3,\dots,2^n-1\}$, we define $x_0=0$ and the probability of the FPT to a continuous piecewise linear boundary g^n on $[0,t_{m-1}^n]$ and with trend $\alpha\in\mathbb{R}$ on $[t_{m-1}^n,t_m^n]$ as $G_m^n:\mathbb{R}\to\mathbb{R}$ such that

$$G_m^n(\alpha) = 1-\sum_{k=1}^{m-1}G_k^n(\alpha_k^n)-\int_{-\infty}^{\alpha_0^n+\alpha_1^n\Delta_n}\int_{-\infty}^{\alpha_0^n+(\alpha_1^n+\alpha_2^n)\Delta_n}\dots\int_{-\infty}^{\alpha_0^n+(\sum_{k=1}^m\alpha_k^n+\alpha)\Delta_n}$$

$$\Big(1-$$

$$\exp\Big(\frac{-2(\alpha_0^n+(\sum_{i=1}^m\alpha_i^n+\alpha)\Delta_n-x_{m+1})(\alpha_0^n+\sum_{i=1}^m\alpha_i^n\Delta_n-x_m)}{\Delta_n}\Big)\Big)$$

$$\times\frac{\exp\big(-\frac{(x_{m+1}-x_m)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}}$$

$$\prod_{k=0}^{m-1}\Big(1-\exp(\frac{-2(\alpha_0^n+\sum_{i=1}^{k+1}\alpha_i^n\Delta_n-x_{k+1})(\alpha_0^n+\sum_{i=1}^k\alpha_i^n\Delta_n-x_k)}{\Delta_n}\Big)\Big)$$

$$\times\frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}}dx_1dx_2\dots dx_m. \tag{31}$$

In the lemma that follows, we give a more explicit form to α_m^n based on known values $(\alpha_k^n)_{k=0,\dots,m-1}$.

Lemma 12. *For any $n \in \mathbb{N}$ and any $m \in \{3, \dots, 2^n - 1\}$, Equation (8) can be reexpressed as*

$$G_m^n(\alpha_m^n) - \int_{t_{m-1}^n}^{t_m^n} f(s)ds = 0. \tag{32}$$

Proof of Lemma 12. We have that

$$\begin{aligned}
\mathbb{P}\big(T_{g^n}^W \in [t_{m-1}^n, t_m^n]\big) &= \mathbb{P}\left((\mathrm{T}_{g^n}^W < t_{m-1}^n, \mathrm{T}_{g^n}^W > t_m^n)^C\right) \\
&= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_{m-1}^n, \mathrm{T}_{g^n}^W > t_m^n\right) \\
&= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_{m-1}^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_m^n\right) \\
&= 1 - \mathbb{P}\left(0 \leq \mathrm{T}_{g^n}^W < t_{m-1}^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_m^n\right) \\
&= 1 - \int_0^{t_{m-1}^n} f(s)ds - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_m^n\right) \\
&= 1 - \sum_{k=1}^{m-1} G_k^n(\alpha_k^n) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_m^n\right). \qquad (33)
\end{aligned}$$

Here, we use elementary probability facts in the first and second equalities, the fact that $\{\mathrm{T}_{g^n}^W < t_{m-1}^n\}$ and $\{\mathrm{T}_{g^n}^W > t_m^n\}$ are disjoint events in the third equality, the fact that $T_{g^n}^W \geq 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the fourth equality, Equation (8) in the fifth equality, Lemmas 10 and 11 in the sixth equality. Also, we have that

$$\begin{aligned}
\mathbb{P}\left(\mathrm{T}_{g^n}^W > t_m^n\right) &= \mathbb{P}\Big(W_{t_1^n} \in (-\infty, g^n(t_1^n)], W_{t_2^n} \in (-\infty, g^n(t_2^n)], \dots, \\
&\quad W_{t_m^n} \in (-\infty, g^n(t_m^n)], \mathrm{T}_{g^n}^W > t_m^n\Big) \\
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} \dots \int_{-\infty}^{g^n(t_m^n)} p_{g^n}^W(t_1^n, x_1, t_2^n, x_2, \dots, \\
&\quad t_m^n, x_m \mid 0, 0) dx_1 dx_2 \dots dx_m \\
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} \dots \int_{-\infty}^{g^n(t_m^n)} \prod_{k=0}^{m-1} p_{g^n}^W(t_{k+1}^n, x_{k+1} \mid t_k^n, x_k) \\
&\quad dx_1 dx_2 \dots dx_m
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{g^n(t_1^n)} \int_{-\infty}^{g^n(t_2^n)} \cdots \int_{-\infty}^{g^n(t_m^n)} \prod_{k=0}^{m-1} \Big(1 - \\
&\quad \exp\Big(\frac{-2(g^n(t_{k+1}^n) - x_{k+1})(g^n(t_k^n) - x_k)}{\Delta_n}\Big)\Big) \\
&\quad \times \frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} dx_1 dx_2 \dots dx_m, \\
&= \int_{-\infty}^{\alpha_0^n+\alpha_1^n\Delta_n} \int_{-\infty}^{\alpha_0^n+(\alpha_1^n+\alpha_2^n)\Delta_n} \cdots \int_{-\infty}^{\alpha_0^n+\sum_{k=1}^m \alpha_k^n\Delta_n} \prod_{k=0}^{m-1} \\
&\quad \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \sum_{i=1}^{k+1} \alpha_i^n\Delta_n - x_{k+1})(\alpha_0^n + \sum_{i=1}^{k} \alpha_i^n\Delta_n - x_k)}{\Delta_n}\Big)\Big) \\
&\quad \times \frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} dx_1 dx_2 \dots dx_m. \qquad (34)
\end{aligned}
$$

Here, we use the fact that

$$
\{\mathrm{T}_{g^n}^W > t_m^n\} \subset \{W_{t_1^n} \in (-\infty, g^n(t_1^n)], W_{t_2^n} \in (-\infty, g^n(t_2^n)], \dots, W_{t_m^n} \in (-\infty, g^n(t_m^n)]\}
$$

in the first equality, Equation (14) in the second equality, Equation (12) in the third equality, Equation (21) from Lemma 8 in the fourth equality and Equations (5)-(6) in the fifth equality. Finally, we can deduce Equation (32) by plugging Equation (34) and Equation (8) into Equation (33). □

With the same arguments as in the proofs of Lemmas 9, 10,11 and 12, for any $n \in \mathbb{N}$ we can define H_0^n as $H_0^n : \mathbb{R}_*^+ \to \mathbb{R}$ and H_m^n as $H_m^n : \mathbb{R} \to \mathbb{R}$ for any $m \in \{1, \dots, 2^n\}$, which are defined in the same way as G_m^n for the reflected Wiener process. As the obtained equations are longer than in the Wiener process case, we do not report them. The next lemma will be useful in showing the existence and unicity of α_0^n, i.e., in the proof of Proposition 2. This basically states that the probability of the FPT started at time t_0^n to a constant boundary on $[t_0^n, t_1^n]$, i.e., G_0^n or H_0^n, is a strictly decreasing bijection from $\mathbb{R}_*^+$ to $(0, 1)$.

Lemma 13. *For any $n \in \mathbb{N}$ we have that G_0^n and H_0^n are continuous and strictly decreasing bijections from $\mathbb{R}_*^+$ to $(0, 1)$.*

Proof. From Equation (22), we can see that G_0^n is differentiable on $\mathbb{R}_*^+$ with negative derivatives for any $n \in \mathbb{N}$. Thus, we have that G_0^n is continuous and strictly decreasing. We also have that $G_0^n(\alpha) \to 1$ as $\alpha \to 0$ and $G_0^n(\alpha) \to 0$ as $\alpha \to \infty$, thus G_0^n is a bijection from $\mathbb{R}_*^+$ to $(0, 1)$. We can prove the case H_0^n with the same arguments. □

The next lemma is the counterpart of Lemma 13 when considering G_m^n and H_m^n for any $n \in \mathbb{N}$ and any $m \in \{1, \dots, 2^n\}$.

Lemma 14. *For any $n \in \mathbb{N}$ and any $m \in \{1, \dots, 2^n\}$, we have that G_m^n and H_m^n are continuous and strictly decreasing bijections from $\mathbb{R}$ to $(0, \int_{t_m^n}^{+\infty} f(s)ds)$.*

Proof. From Equations (25), (28) and (31), we can see that G^n_m is differentiable on $\mathbb{R}$ with negative derivative for any $n \in \mathbb{N}$ and any $m \in \{1, \dots, 2^n\}$. Thus, we have that G^n_m is continuous and strictly decreasing. We also have that $G^n_m(\alpha) \to \int_{t^n_m}^{+\infty} f(s)ds$ as $\alpha \to -\infty$ and $G^n_m(\alpha) \to 0$ as $\alpha \to \infty$, thus G^n_m is continuous a bijection $\mathbb{R}$ to $(0, \int_{t^n_m}^{+\infty} f(s)ds)$. We can prove the case H^n_m with the same arguments. □

The following lemma shows the positivity of the integral of f between two approximation times when we assume that Assumption 1 holds.

Lemma 15. *We assume that Assumption 1 holds. Then, we have for any $n \in \mathbb{N}$ that*

$$0 < \int_0^{\delta_n \Delta_n} f(s)ds. \tag{35}$$

For any $n \in \mathbb{N}$ and any $m \in \{1, \dots, 2^n\}$, we also have that

$$0 < \int_{t^n_{m-1}}^{t^n_m} f(s)ds. \tag{36}$$

Proof of Lemma 15. We define the supremum of the boundary absolute value g on $[0, t_f]$ as

$$g_+ = \sup_{t \in [0, t_f]} \mid g(t) \mid .$$

By Assumption 1, we have that g is continuous on $[0, t_f]$. Since $[0, t_f]$ is a compact space, it implies that $g_+ < \infty$. We have that $g \in \mathcal{G}$, thus $g_+ > 0$ by Definition 1. By Definition 2, we can deduce that $\mathrm{T}^Z_g \leq \mathrm{T}^Z_{g_+}$ a.s.. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}^Z_{g_+} \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_g \in [0, \delta_n \Delta_n]). \tag{37}$$

Since $\mathbb{P}(\mathrm{T}^W_{g_+} \in [0, \delta_n \Delta_n]) = G^n_0(g_+)$ and $\mathbb{P}(\mathrm{T}^{|W|}_{g_+} \in [0, \delta_n \Delta_n]) = H^n_0(g_+)$, we obtain by Lemma 13 that $\mathbb{P}(\mathrm{T}^Z_{g_+} \in [0, \delta_n \Delta_n]) > 0$. Then, we can deduce Equation (35) since f is equal to the density of T^Z_g by Equation (2) and Equation (4). Equation (36) follows by induction on $m \in \{1, \dots, 2^n\}$ with similar arguments. □

We give now the proof of Proposition 2.

Proof of Proposition 2. For any $n \in \mathbb{N}$, we prove Proposition 2 by induction on $m \in \{0, \dots, 2^n\}$. We start with the $m = 0$ case, i.e., we show that $\alpha^n_0 \in \mathbb{R}^+_*$ is well-defined. By Lemma 15 along with Assumption 1 we can deduce that

$$0 < \int_0^{\delta_n \Delta_n} f(s)ds < 1. \tag{38}$$

From Expression (38) and Lemma 9, we can then deduce that

$$0 < G_0^n(\alpha_0^n) < 1 \text{ and } 0 < H_0^n(\alpha_0^n) < 1. \tag{39}$$

Finally, an application of the intermediate value theorem together with Lemma 13 and Expression (39) provides the existence and uniqueness of $\alpha_0^n \in \mathbb{R}_*^+$. We consider now the $m > 0$ case, i.e., we show that $\alpha_m^n \in \mathbb{R}$ is well-defined. By Lemma 15 along with Assumption 1 we get

$$0 < \int_{t_{m-1}^n}^{t_m^n} f(s)ds < \int_{t_{m-1}^n}^{+\infty} f(s)ds. \tag{40}$$

From Expression (40), Lemmas 10, 11 and 12, we can deduce that

$$0 < G_m^n(\alpha_m^n) < \int_{t_{m-1}^n}^{+\infty} f(s)ds \text{ and } 0 < H_m^n(\alpha_m^n) < \int_{t_{m-1}^n}^{+\infty} f(s)ds. \tag{41}$$

To conclude, an application of the intermediate value theorem along with Lemma 14 and Equation (41) provides the existence and uniqueness of $\alpha_m^n \in \mathbb{R}$. ☐

6. Proofs of the main results

In this section, we first show that the approximation converges to the boundary starting value, i.e., Proposition 3. The elementary idea of the proof consists in observing that the boundary can be bounded below and above by positive constants for a very small time.

Proof of Proposition 3. For any $n \in \mathbb{N}$, we define the infimum and the supremum of the boundary g on $[0, \delta_n\Delta_n]$ as

$$g_-^n(0) = \inf_{t\in[0,\delta_n\Delta_n]} g(t) \ , \ g_+^n(0) = \sup_{t\in[0,\delta_n\Delta_n]} g(t).$$

By Assumption 1, we have that g is continuous on $[0, \delta_n\Delta_n]$. Since $[0, \delta_n\Delta_n]$ is a compact space, it implies that $-\infty < g_-^n(0) \leq g_+^n(0) < \infty$ for any $n \in \mathbb{N}$. We have that $g \in \mathcal{G}$, thus $g_+^n(0) > 0$ for any $n \in \mathbb{N}$ and $g_-^n(0) > 0$ for n big enough by Definition 1. By Definition 2, we can deduce that $\mathrm{T}^Z_{g_-^n(0)} \leq \mathrm{T}^Z_g \leq \mathrm{T}^Z_{g_+^n(0)}$ a.s. and for n big enough. Thus, we can deduce for n big enough that

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n\Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_g \in [0, \delta_n\Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n\Delta_n]).$$

By Equations (2) and (3), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n\Delta_n]) \leq \int_0^{\delta_n\Delta_n} f_g^Z(s)ds \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n\Delta_n]).$$

By Equation (4), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n\Delta_n]) \leq \int_0^{\delta_n\Delta_n} f(s)ds \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n\Delta_n]).$$

By Equation (7), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^{Z}_{g^n_+(0)} \in [0, \delta_n\Delta_n]) \leq \mathbb{P}(\mathrm{T}^{Z}_{\alpha^n_0} \in [0, \delta_n\Delta_n]) \leq \mathbb{P}(\mathrm{T}^{Z}_{g^n_-(0)} \in [0, \delta_n\Delta_n]).$$

By Equation (22) and Lemma 9, the above inequalities can be reexpressed as

$$G^n_0(g^n_+(0)) \leq G^n_0(\alpha^n_0) \leq G^n_0(g^n_-(0)) \text{ or } H^n_0(g^n_+(0)) \leq H^n_0(\alpha^n_0) \leq H^n_0(g^n_-(0)). \quad (42)$$

Since we have that G^n_0 and H^n_0 are continuous and strictly decreasing bijections from $\mathbb{R}^+_*$ to $(0, 1)$ by Lemma 13, we can invert the six sides of Expression (42) by G^n_0 when $Z = W$ and H^n_0 when $Z = |W|$. We obtain that $g^n_-(0) \leq \alpha^n_0 \leq g^n_+(0)$. By continuity of g on $[0, t_f]$ and Equation (5), we can conclude $\alpha^n_0 = g^n(0) \to g(0)$ as $n \to \infty$.

□

Now, we show that a subsequence of the approximation uniformly converges to the boundary when the length of each interval of linear approximation goes to 0 asymptotically. The proof goes in two steps. First, we show that the approximation uniformly converges to some boundary $\tilde{g} \in \mathcal{G}$ using Arzelà-Ascoli theorem on any compact space $[0, t_f]$. Second, we show that $\tilde{g}(t) = g(t)$ for any $t \in [0, t_f]$. We first give the definition of the piecewise linear boundary functions.

Definition 5. For any $n \in \mathbb{N}$, we define the set of piecewise linear boundary functions as

$$\mathcal{G}^n = \Big\{ g \in \mathcal{G} \text{ s.t. } g \text{ is linear on each interval } \big[t^n_m, t^n_{m+1}\big] \text{ for any } m \in \{0, \dots, 2^n\} \Big\}.$$

In what follows, we give the definition of uniform boundedness.

Definition 6. The sequence $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly bounded if there is a constant $M > 0$ such that

$$\sup_{t \in [0, t_f], n \in \mathbb{N}} |g^n(t)| \leq M. \quad (43)$$

The following definition introduces the notion of uniform equicontinuity.

Definition 7. The sequence $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly equicontinuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{t, s \in [0, t_f], |t-s| < \delta, n \in \mathbb{N}} ||g^n(t) - g^n(s)|| \leq \varepsilon. \quad (44)$$

We give now the Arzelà-Ascoli theorem.

Theorem 16 (Arzelà-Ascoli theorem)**.** *If the sequence $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly bounded and uniformly equicontinuous, then there exists a subsequence which converges uniformly to some $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$.*

In the following proposition, we show that if we assume that the α^n_m are uniformly bounded, then the sequence g^n is uniformly bounded and uniformly equicontinuous.

Proposition 17. *We assume that Assumption 1 holds and that the α_m^n are uniformly bounded, i.e.,*

$$\sup_{\substack{n\in\mathbb{N}\\ m=0,\cdots,2^n}} |\alpha_m^n| \leq K. \tag{45}$$

Then, the sequence g^n is uniformly bounded and uniformly equicontinuous, i.e., it satisfies Equations (43) and (44).

Proof. We start with the proof of Equation (43). By algebraic manipulation, we can rewrite Equations (5) and (6) for $m \in \{1, \ldots, 2^n - 1\}$ as

$$g^n(u) \quad = \quad \alpha_0^n + \Delta_n \sum_{i=1}^{m} \alpha_i^n + \alpha_{m+1}^n(u - t_m^n), u \in (t_m^n, t_{m+1}^n]. \tag{46}$$

We obtain that for $u \in (t_m^n, t_{m+1}^n]$ where $m \in \{1, \ldots, 2^n - 1\}$ that

$$\begin{aligned}
|g^n(u)| &\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{m} |\alpha_i^n| + \left|\alpha_{m+1}^n\right|(u - t_m^n) \\
&\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{m+1} |\alpha_i^n| \\
&\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{2^n} |\alpha_i^n| \\
&\leq |\alpha_0^n| + t_f \sup_{\substack{n\in\mathbb{N}\\ i=1,\cdots,2^n}} |\alpha_i^n| \\
&\leq (1 + t_f) \sup_{\substack{n\in\mathbb{N}\\ i=0,\cdots,2^n}} |\alpha_i^n| \\
&\leq (1 + t_f)K.
\end{aligned}$$

Here, we use the triangular inequality in the first inequality, the fact that $u \in (t_m^n, t_{m+1}^n]$ in the second inequality, the fact that $m \in \{1, \ldots, 2^n - 1\}$ in the third equality, the definition of Δ_n in the fourth equality, and Equation (45) in the last inequality. We have thus shown that Equation (45) $\implies$ Equation (43). We now prove Equation (44). We consider any arbitrarily small $\varepsilon > 0$. Accordingly, we set

$$\delta = \frac{\varepsilon}{2K}. \tag{47}$$

For any $t \in [0, t_f]$, we define the corresponding m_t^n such that $t \in [t_{m_t^n}^n, t_{m_t^n+1}^n]$. From Equation (46), we can deduce that

$$g^n(t) \quad = \quad \alpha_0^n + \Delta_n \sum_{i=1}^{m_t^n} \alpha_i^n + \alpha_{m_t^n+1}^n(t - t_{m_t^n}^n). \tag{48}$$

Thus, for any $0 \leq s \leq t \leq t_f$ such that

$$|t - s| < \delta, \tag{49}$$

we have that

$$\begin{aligned}
|g^n(t) - g^n(s)| &= \Big|\alpha_0^n + \Delta_n \sum_{i=1}^{m_t^n} \alpha_i^n + \alpha_{m_t^n+1}^n (t - t_{m_t^n}^n) - \\
&\quad (\alpha_0^n + \Delta_n \sum_{i=1}^{m_s^n} \alpha_i^n + \alpha_{m_s^n+1}^n (s - t_{m_s^n}^n))\Big| \\
&= \Big|\Delta_n \sum_{i=m_s^n}^{m_t^n} \alpha_i^n + \alpha_{m_t^n+1}^n (t - t_{m_t^n}^n) - \alpha_{m_s^n+1}^n (s - t_{m_s^n}^n)\Big| \\
&\leq |t - s| \sup_{\substack{n \in \mathbb{N} \\ i=0,\cdots,2^n}} |\alpha_i^n| \\
&\leq K\,|t - s|\,, \\
&\leq \varepsilon.
\end{aligned}$$

Here, we use Equation (48) in the first equality, algebraic manipulation in the second equality and the first equality, Equation (45) in the second inequality, Equation (47) and Expression (49) in the last inequality. We have thus shown that Equation (45) $\implies$ Equation (44). □

In the following proposition, we show that if we assume that Assumption 2 holds, then we have that the α_m^n are uniformly bounded.

Proposition 18. *We assume that Assumption 2 holds. Then, we have that the α_m^n are uniformly bounded, i.e., Equation (45) is satisfied.*

Proof. We define the supremum of the boundary derivative absolute value g' on $[0, t_f]$ as

$$g'_+ = \sup_{t \in [0, t_f]} |\, g'(t) \,|\,.$$

We define the bound, which does not depend on n or m, as

$$K = 2 \sup(g_+, g'_+). \tag{50}$$

By Assumption 2, we have that g is continuous on $[0, t_f]$. Since $[0, t_f]$ is a compact space, it implies that $g_+ < \infty$. We can also obtain by Assumption 2 that $g'_+ < \infty$. Thus, we can deduce that $K < \infty$. Then, to prove Proposition 18 it is sufficient to show that Equation (45) is satisfied with K defined in Equation (50). For any $n \in \mathbb{N}$, we consider a proof by induction on $m \in \{0, \ldots, 2^n\}$. We start with the case $m = 0$, i.e., we show that $\alpha_0^n \leq K$. We have that $g \in \mathcal{G}$, thus $g_+ > 0$ by Definition 1. By Definition 2, we can deduce that $\mathrm{T}_g^Z \leq \mathrm{T}_{g_+}^Z$ a.s.. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}_g^Z \in [0, \delta_n \Delta_n]).$$

By Equations (2) and (3), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+} \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f^Z_g(s) ds.$$

By Equation (4), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+} \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f(s) ds.$$

By Equation (7), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+} \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_{\alpha_0^n} \in [0, \delta_n \Delta_n]).$$

By Equation (22) and Lemma 9, the above inequality can be reexpressed as

$$G_0^n(g_+) \leq G_0^n(\alpha_0^n) \text{ or } H_0^n(g_+) \leq H_0^n(\alpha_0^n). \tag{51}$$

Since we have that G_0^n and H_0^n are continuous and strictly decreasing bijections from $\mathbb{R}_*^+$ to $(0, 1)$ by Lemma 13, we can invert the four sides of Expression (51) by G_0^n when $Z = W$ and H_0^n when $Z = |W|$. We can deduce that $\alpha_0^n \leq g_+$ which implies that $\alpha_0^n \leq K$.

We consider now the case $m = 1$, i.e., we show that $| \alpha_1^n | \leq K$. For $t \geq 0$, we define the linear boundary started at $g(0)$ with trend g'_+ and $-g'_+$ as respectively $\overline{g}(t) = g(0) + g'_+ t$ and $\underline{g}(t) = g(0) - g'_+ t$. By Definition 2, we can deduce that $\mathrm{T}^Z_{\underline{g}} \leq \mathrm{T}^Z_g \leq \mathrm{T}^Z_{\overline{g}}$ a.s.. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}^Z_{\overline{g}} \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}^Z_g \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}^Z_{\underline{g}} \in [0, t_1^n]).$$

For $t \geq 0$, we define now the linear boundary started at α_0^n with trend K and $-K$ as respectively $\overline{g}^n(t) = \alpha_0^n + Kt$ and $\underline{g}^n(t) = \alpha_0^n - Kt$. If n is big enough, we obtain that

$$\mathbb{P}(\mathrm{T}^Z_{\overline{g}^n} \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}^Z_{g^n} \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}^Z_{\underline{g}^n} \in [0, t_1^n]).$$

By Equation (25) and Lemma 10, the above inequalities can be reexpressed as

$$G_1^n(K) \leq G_1^n(\alpha_1^n) \leq G_1^n(-K) \text{ or } H_1^n(K) \leq H_1^n(\alpha_1^n) \leq H_1^n(-K). \tag{52}$$

Since by Lemma 14 we have that G_1^n and H_1^n are continuous and strictly decreasing bijections from $\mathbb{R}$ to $(0, \int_{t_1^n}^{+\infty} f(s) ds)$, we can invert the six sides of Expression (52) by G_1^n when $Z = W$ and H_1^n when $Z = |W|$. We can deduce that $| \alpha_1^n | \leq g'_+$ which implies $| \alpha_1^n | \leq K$.

We consider now the case $m = 2$, i.e., we show that $| \alpha_2^n | \leq K$. We define the boundary which is equal to g on $[0, t_1^n]$ and linear with trend g'_+ for $t \geq t_1^n$ as $\overline{g}(t) = g(t)$ for any $t \in [0, t_1^n]$ and $\overline{g}(t) = g(t_1^n) + g'_+(t - t_1^n)$ for any $t \geq t_1^n$. We also define the boundary which is equal to g on $[0, t_1^n]$ and linear with trend $-g'_+$ for $t \geq t_1^n$ as $\underline{g}(t) = g(t)$ for any $t \in [0, t_1^n]$ and $\underline{g}(t) = g(t_1^n) - g'_+(t - t_1^n)$ for any $t \geq t_1^n$. By Definition 2, we can deduce that $\mathrm{T}_{\underline{g}}^Z \leq \mathrm{T}_g^Z \leq \mathrm{T}_{\overline{g}}^Z$ a.s.. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}_{\overline{g}}^Z \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}_g^Z \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}_{\underline{g}}^Z \in [t_1^n, t_2^n]).$$

We define the boundary which is equal to g^n on $[0, t_1^n]$ and linear with trend K for $t \geq t_1^n$ as $\overline{g}^n(t) = g^n(t)$ for any $t \in [0, t_1^n]$ and $\overline{g}(t) = g(t_1^n) + K(t - t_1^n)$ for any $t \geq t_1^n$. We also define the boundary which is equal to g^n on $[0, t_1^n]$ and linear with trend $-K$ for $t \geq t_1^n$ as $\underline{g}^n(t) = g^n(t)$ for any $t \in [0, t_1^n]$ and $\underline{g}^n(t) = g^n(t_1^n) - K(t - t_1^n)$ for any $t \geq t_1^n$. If n is big enough, we obtain that

$$\mathbb{P}(\mathrm{T}_{\overline{g}^n}^Z \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}_{g^n}^Z \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}_{\underline{g}^n}^Z \in [t_1^n, t_2^n]).$$

By Equation (28) and Lemma 11, the above inequalities can be reexpressed as

$$G_2^n(K) \leq G_2^n(\alpha_2^n) \leq G_2^n(-K) \text{ or } H_2^n(K) \leq H_2^n(\alpha_2^n) \leq H_2^n(-K). \tag{53}$$

Since by Lemma 14 we have that G_2^n and H_2^n are continuous and strictly decreasing bijections from $\mathbb{R}$ to $(0, \int_{t_2^n}^{+\infty} f(s)ds)$, we can invert the six sides of Expression (53) by G_2^n when $Z = W$ and H_2^n when $Z = |W|$. We can deduce that $\mid \alpha_2^n \mid \leq K$. The case with $m > 2$ follows with similar arguments. □

The following corollary is an application of Arzelà-Ascoli theorem.

Corollary 19. *We assume that Assumption 2 holds. Then, there exists a subsequence g^{n_k} of g^n which converges uniformly to some $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$.*

Proof. This is an application of Theorem 16 along with Propositions 17 and 18. □

The following lemma gives a.s. convergence of $T_{h^n}^Z \mathbf{1}_{\{T_{h^n}^Z \leq t_f\}}$ to $T_h^Z \mathbf{1}_{\{T_h^Z \leq t_f\}}$ when h^n converges uniformly to h on $[0, t_f]$. For the proof of Theorem 4, we only need the convergence in distribution.

Lemma 20. *For any sequence $h^n \in \mathcal{G}^n$ which converges uniformly on $[0, t_f]$ to some $h \in \mathcal{G}$ satisfying Assumption 1, we have that $\mathrm{T}_{h^n}^Z \mathbf{1}_{\{\mathrm{T}_{h^n}^Z \leq t_f\}}$ converges a.s. to $\mathrm{T}_h^Z \mathbf{1}_{\{\mathrm{T}_h^Z \leq t_f\}}$. As a by-product, we deduce that $\mathrm{T}_{h^n}^Z$ converges in distribution to T_h^Z on $[0, t_f]$.*

Proof. To prove that $\mathrm{T}_{h^n}^Z \mathbf{1}_{\{\mathrm{T}_{h^n}^Z \leq t_f\}}$ converges a.s. to $\mathrm{T}_h^Z \mathbf{1}_{\{\mathrm{T}_h^Z \leq t_f\}}$, it is sufficient to show that for any arbitrarily small $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}_*$ with $n \geq N_\epsilon$ we have a.s.

$$\left| \mathrm{T}_{h^n}^Z \mathbf{1}_{\{\mathrm{T}_{h^n}^Z \leq t_f\}} - \mathrm{T}_h^Z \mathbf{1}_{\{\mathrm{T}_h^Z \leq t_f\}} \right| \leq \epsilon. \tag{54}$$

As h^n converges uniformly to h on $[0, t_f]$, we have that for any $\epsilon_h > 0$, there exists $N_{\epsilon_h} \in \mathbb{N}$ such that for any $n \in \mathbb{N}_*$ with $n \geq N_{\epsilon_h}$ we have

$$\sup_{t \in [0, t_f]} \mid h^n(t) - h(t) \mid \leq \epsilon_h. \tag{55}$$

We set the value of ϵ_h as

$$\epsilon_h = \frac{1}{2} \sup_{\mathrm{T}_h^Z \leq t \leq \mathrm{T}_h^Z + \epsilon \leq t_f} |Z_t - h(t)| \,. \tag{56}$$

First, we can see that ϵ_h defined in Equation (56) is positive. Second, we have that a.s. Z_t first hits h^n on $[\mathrm{T}_h^Z - \epsilon, T_h^Z + \epsilon]$, i.e., we have shown that $\mathrm{T}_{h^n}^Z \in [\mathrm{T}_h^Z - \epsilon, \mathrm{T}_h^Z + \epsilon]$ whenever Expression (55) holds with ϵ_h from Equation (56). Thus, we have shown Expression (54) with $N_\epsilon = N_{\epsilon_h}$. □

We consider a discretization length in the order Δ_n so that we obtain that the time discretization is nested, i.e., for any t_m^n and any $l \geq m$ there exists a time t_k^l such that $t_m^n = t_k^l$. This is required to prove the following lemma which in turn will be used to prove that the limit of a subsequence obtained by Arzelà-Ascoli theorem satisfies Equation (4).

Lemma 21. *We assume that Assumption 1 holds. For any $n \in \mathbb{N}$, any $l \in \mathbb{N}$ with $l \geq n$ and any $m \in \{0, \dots, 2^n\}$, the approximated boundary satisfies*

$$\mathbb{P}\left(\mathrm{T}_{g^l}^Z \in [t_m^n, t_{m+1}^n]\right) = \int_{t_m^n}^{t_{m+1}^n} f(s) ds. \tag{57}$$

Proof. For any $n \in \mathbb{N}$, any $l \in \mathbb{N}$ with $l \geq n$ and any $m \in \{0, \dots, 2^n\}$, we have

$$\begin{aligned}
\mathbb{P}\left(\mathrm{T}_{g^l}^Z \in [t_m^n, t_{m+1}^n]\right) &= \sum_{\substack{i \in \mathbb{N} \text{ s.t.} \\ t_m^n \leq t_i^l \leq t_{i+1}^l \leq t_{m+1}^n}} \mathbb{P}\left(\mathrm{T}_{g^l}^Z \in [t_i^l, t_{i+1}^l]\right) \\
&= \sum_{\substack{i \in \mathbb{N} \text{ s.t.} \\ t_m^n \leq t_i^l \leq t_{i+1}^l \leq t_{m+1}^n}} \int_{t_i^l}^{t_{i+1}^l} f(s) ds \\
&= \int_{t_m^n}^{t_{m+1}^n} f(s) ds.
\end{aligned}$$

Here, we use the fact that

$$[t_m^n, t_{m+1}^n] = \bigcup_{\substack{i \in \mathbb{N} \text{ s.t.} \\ t_m^n \leq t_i^l \leq t_{i+1}^l \leq t_{m+1}^n}} [t_i^l, t_{i+1}^l]$$

since the time discretization is nested in the first equality, and Equations (7) and (8) in the second equality. □

We provide in what follows the proof of the main result of our paper, which shows that a subsequence of the new approximation uniformly converges to the boundary when the length of each interval of linear approximation goes to 0 asymptotically. This proof is based on application of previously obtained results and shows that $\tilde{g}(t) = g(t)$ for any $t \in [0, t_f]$.

Proof of Theorem 4. By Corollary 19 along with Assumption 2, there exists a subsequence g^{n_k} of g^n which converges uniformly to some $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$. We first show that the density of $f^Z_{\tilde{g}}(t) = f(t)$ for any $t \in [0, t_f]$. By Borel arguments, it is sufficient to show that for any $p \in \mathbb{N}$ and any $k \in \{0, \ldots, 2^p - 1\}$ we have

$$\mathbb{P}\big(\mathrm{T}^Z_{\tilde{g}} \in [k\Delta_p, (k+1)\Delta_p]\big) = \int_{k\Delta_p}^{(k+1)\Delta_p} f(s)ds. \tag{58}$$

We have that

$$\begin{aligned}\mathbb{P}\big(\mathrm{T}^Z_{\tilde{g}} \in [k\Delta_p, (k+1)\Delta_p]\big) &= \lim_{n\to\infty} \mathbb{P}\big(\mathrm{T}^Z_{g^{n_k}} \in [k\Delta_p, (k+1)\Delta_p]\big) \\ &= \int_{k\Delta_p}^{(k+1)\Delta_p} f(s)ds.\end{aligned}$$

Here, the first equality corresponds to the convergence in distribution of $\mathrm{T}^Z_{g^{n_k}}$ to $\mathrm{T}^Z_{\tilde{g}}$ by Lemma 20 along with Assumption 2, and we use Lemma 21 along with Assumption 2 in the second equality. Thus, we have shown Equation (58), which implies that $f^Z_{\tilde{g}}(t) = f(t)$ for any $t \in [0, t_f]$. Since there is uniqueness of the IFPT problem, we can deduce that $\tilde{g}(t) = g(t)$ for any $t \in [0, t_f]$. □

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