On convergence of piecewise-linear approximation in the inverse first-passage time problem

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Abstract: The inverse first-passage time problem determines a boundary function such that the first-passage time of a Wiener process to this boundary function has a given distribution. An approximation of the boundary function by a piecewise-linear boundary is given by equating the probability of the first-passage time to a linear boundary and the increment of the distribution on each interval. This is based on the starting value of the boundary function, which is unknown in practice. We propose an approximation for the starting value of the boundary function. We consider asymptotics where the length of each interval goes to 0. We first show that the approximation for the starting value of the boundary function converges to the starting value of the boundary function when assuming that the boundary function is absolutely continuous and with positive starting value. We also show that a subsequence of the piecewise-linear approximation uniformly converges to the boundary function. The proofs are based on an application of Arzelà-Ascoli theorem. A numerical study shows that the piecewise-linear approximation is sensitive to the starting value of the boundary function and the starting value of the boundary function derivative. The results obtained in the numerical study indicate that the piecewise-linear approximation is adequate and relatively safe to use in practice.

Keywords and phrases: Applied probability, inverse first-passage time problem, piecewise-linear approximation, boundary function, convergence.

1. Introduction

This paper concerns the inverse first-passage time (IFPT) problem. The IFPT problem determines the boundary function such that the first-passage time (FPT) of a standard Wiener process to this boundary function has a given distribution. This problem was formulated by A. Shiryaev during a Banach center meeting in 1976. More specifically, he considered the particular case of exponential distribution, which is commonly referred as the inverse Shiryaev problem. The IFPT is an important problem in applied probability.

The primary application of the IFPT problem is in portfolio credit risk modeling. Initially, the focus was on random walks (see Iscoe, Kreinin and Rosen (1999)). A detailed analysis of the IFPT problem and an approximation is given in Iscoe and Kreinin (1999). A model of default events with a randomized boundary is proposed in Schmidt and Novikov (2008). Another field of application is in pricing of credit derivatives (see Avellaneda and Zhu (2001) and Hull and White (2001)). The stochastic process represents the so-called distance to default of an obligor, while the FPT represents a default event. The boundary function stands for a barrier separating the healthy states of the obligor from the default state. Another application is in inventory-control problem, whose formulation can be equivalent to the IFPT problem (see Song and Zipkin (2013)).

Despite their importance for applications, explicit solutions of the IFPT are very limited. There are some examples when the boundary function is linear. Doob (1949) gives explicit formulae of crossing boundary probabilities (see Equations (4.2)-(4.3), pp. 397-398) based on elementary geometrical and analytical arguments. Malmquist (1954) obtains an explicit formula conditioned on the starting and final values of the Wiener process (Theorem 1, p. 526). This is obtained with Doob's transformation (see Section 5, pp. 401-402). Anderson (1960) derives an explicit formula conditioned on the final value of the Wiener process (see Theorem 4.2, pp. 178-179). Then, he integrates it with respect to the final value of the Wiener process to get an explicit solution (see Theorem 4.3, p. 180). For square root boundaries, Breiman (1967) expresses the FPT problem as an FPT of an Ornstein-Uhlenbeck process to a constant boundary. They are obtained with Doob's transformation. However, the boundary crossing probabilities of an Ornstein-Uhlenbeck process to a constant boundary are only known in the form of a Laplace transform. Daniels (1969) uses the same technique and obtains an explicit solution. More recently, Potiron (2025a) obtains non-explicit formulae by the Girsanov theorem. Finally, Potiron (2025b) obtains an explicit formula when the boundary function is constant and the stochastic process is a continuous local martingale.

As explicit solutions are very limited, the literature related to the IFPT problem relies heavily on approximations (see Zucca and Sacerdote (2009) and Song and Zipkin (2011)). In Zucca and Sacerdote (2009), an approximation to a continuous boundary function by a piecewise-linear boundary is given by equating the probability of the FPT to a linear boundary and the increment of the cumulative distribution function (cdf) on each interval. That piecewise-liner approximation uses Wang and Pötzelberger (1997) idea. That piecewise-linear approximation is based on the starting value of the boundary function, which has to be guessed in practice since it is unknown.

We propose an approximation for the starting value of the boundary function, which makes it more suitable for applications. The idea is to equate the probability of the FPT to a constant boundary function and the increment of the cdf on a first interval. First, we show that the approximation for the starting value of the boundary function converges to the starting value of the boundary function when assuming that the boundary function is absolutely continuous and with positive starting value. Second, we show that a subsequence of the piecewise-linear approximation uniformly converges to the boundary function when assuming that the boundary is differentiable with uniformly bounded derivative. The results are obtained using Arzelà-Ascoli theorem on any compact space.

We consider asymptotics where the length of each interval of linear approximation goes to 0. These asymptotics are required to show that the piecewise-linear approximation goes to the boundary function asymptotically. The use of these asymptotics and the convergence results are new to the literature on the IFPT problem. They are important in practice, although we only obtain the uniform convergence for a subsequence of piecewise-linear approximation. A numerical study shows that the piecewise-linear approximation is sensitive to the starting value of the boundary function and the starting value of the boundary function derivative. The results obtained in the numerical study indicate that the piecewise-linear approximation is adequate and relatively safe to use in practice. The numerical study also illustrates that these asymptotics are adapted to obtain an adequate piecewise-linear approximation.

Since the formulation of the IFPT problem, many papers have investigated its theoretical properties. Dudley and Gutmann (1977) show the existence of a stopping time with respect to a general stochastic process, but this stopping time is not a FPT. The existence of lower semi-continuous solutions was established in Anulova (1981) for the FPT of a reflected Wiener process by compacity arguments in a discrete approximation of the boundary function and the distribution. The IFPT problem is reformulated as a nonlinear Volterra integral equation in Peskir (2002a). Peskir (2002b) studies the behavior of the IFPT problem in the neighborhood of 0. Abundo (2006) consider extensions to the general diffusion process case. When the distribution is non-atomic, Cheng et al. (2006) and Chen et al. (2011) show the existence and uniqueness of the IFPT problem for diffusions by a transfer into a free boundary problem.

More recently, Jaimungal, Kreinin and Valov (2014) consider a connection between the Skorokhod embedding problem and the IFPT problem. For a general distribution, Ekström and Janson (2016) show the existence and uniqueness for Wiener processes by discretizing an optimal stopping problem. Beiglböck et al. (2018) consider a more general optimal stopping problem which yields existence and uniqueness as a consequence. Fukasawa and Obloj (2020) consider efficient discretization of stochastic differential equations based on the FPT of spheres. Chen, Chadam and Saunders (2022) study higher-order regularity properties of the solution of the IFPT problem. The uniqueness for reflected Wiener processes is shown by a discrete approximation argument along with stochastic ordering in Klump and Kolb (2023). Klump and Kolb (2024) prove existence and uniqueness of the IFPT problem for soft-killed Wiener processes. The existence and the uniqueness for Levy processes and diffusions are studied in Klump and Savov (2025).

Our theoretical results also complement the theoretical results on continuous boundary functions obtained in Chen et al. (2011) (see Proposition 6) and Ekström and Janson (2016) (see Theorem 8.2). Compared to these two papers, our approach based on compacity requires stronger assumptions. This is a price to pay as our approach is more direct and circumvents the use of a free boundary problem or optimal stopping theory. The results are also proved in the FPT problem of a reflected Wiener process, which are also novel.

One limitation in this paper is that we only obtain the convergence for a subsequence of the piecewise-linear approximation. The reason is that we are not able to prove directly the convergence by extending the proving techniques used in Zucca and Sacerdote (2009). More specifically, their two main ideas are the use of concavity inequalities and the implicit function theorem. Assuming that the FPT cdf is absolutely continuous and the boundary is monotone concave, they prove in Theorem 4.3 (p. 1331) that the error due to the approximation is of the order equal to the maximum of the initial error and the squared interval length. We can weaken their assumptions on concavity by assumptions on differentiability with uniformly bounded derivatives. The elementary idea of the proof consists in bounding the difference between the approximation and the boundary value by a linear function on each interval. However, we are not able to extend their direct use of the implicit function theorem with the new asymptotics. The reason is that we need to use the implicit function theorem with an increasing number of intervals, whereas they only use it with a finite number of intervals. Although we were not able to track down the calculation, we conjecture that the direct convergence also holds.

The remaining of this paper is structured as follows. We give preliminary results in Section 2. Then, we provide the main results in Section 3. In Section 4, we conduct a numerical study. We establish the proofs from the preliminary results in Section 5. Section 6 yields the proofs of the main results. Finally, we give concluding remarks in Section 7.

2. Preliminary results

We first introduce the probabilistic tools. We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F}, \mathbf{F})$, where $\mathcal{F}$ is a σ-field and $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is a filtration. For any set $A \subset \mathbb{R}^+$ and any set $B \subset \mathbb{R}$ such that $0 \in A$, we define the set of continuous functions with positive starting values as

$$\mathcal{C}_0^+(A, B) = \{h : A \to B \text{ s.t. } h \text{ is continuous and } h(0) > 0\}.$$

We now give the definition of the set of boundary functions. Since the approximation by a piecewise linear boundary given in Wang and Pötzelberger (1997) requires continuity of the boundary, we restrict ourselves to the continuous boundary case. Moreover, we do not allow for a boundary starting from the origin since our techniques unfortunately do not allow for that more complicated case.

Definition 1. We define the set of boundary functions as $\mathcal{G} = \mathcal{C}_0^+(\mathbb{R}^+, \mathbb{R})$.

Then, we give the definition of the FPT. We introduce an $\mathbf{F}$-adapted continuous process Z_t defined for any time $t \in \mathbb{R}^+$. We also introduce a boundary function $g \in \mathcal{G}$. We assume that the stochastic process is continuous since we consider a Wiener process or a reflected Wiener process in this paper.

Definition 2. We define the FPT of the stochastic process Z to the boundary function g as

$$\mathrm{T}_g^Z = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g(t)\}. \tag{1}$$

We define an $\mathbf{F}$-standard Wiener process as W_t defined for any time $t \in \mathbb{R}^+$. We will consider the two cases in the following of this paper:

1. Wiener process: $Z_t = W_t$.
2. reflected Wiener process: $Z_t = |W_t|$.

First, we have that Z is a continuous and $\mathbf{F}$-adapted stochastic process and $\inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g(t)\} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } (t, Z_t) \in G\}$, where $G = \{(t,u) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } u \geq g(t)\}$ is a closed subset of $\mathbb{R}^2$. Thus, the FPT T_g^Z is an $\mathbf{F}$-stopping time by Theorem I.1.27 (p. 7) in Jacod and Shiryaev (2003).

We define the cdf of Z as

$$P_g^Z(t) = \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for any time } t \geq 0. \tag{2}$$

The basic assumption for the approximation by a piecewise linear boundary given in Zucca and Sacerdote (2009) is that the cdf P_g^Z is absolutely continuous. Accordingly, the authors assume that all regularity assumptions ensuring the existence of the objects introduced and properties imposed are fulfilled. In the following assumption, we consider a slightly more explicit form of the assumption.

Assumption 1. We assume that the boundary function g is absolutely continuous on the set of nonnegative real numbers $\mathbb{R}^+$.

When the boundary function g is continuous, we know by Theorem 8.1 in Ekström and Janson (2016) that P_g^Z is continuous. When the boundary function g is continuously differentiable, we know by Lemma 3.3 in Strassen (1967) that the cdf P_g^Z is continuously differentiable. The following lemma shows that when the boundary function g is absolutely continuous, then the cdf P_g^Z is absolutely continuous.

Lemma 1. *We assume that Assumption 1 holds. Then, the cdf P_g^Z is absolutely continuous on the set of nonnegative real numbers $\mathbb{R}^+$.*

Since the cdf P_g^Z is absolutely continuous, there exists a pdf $f_g^Z : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$f_g^Z(t) = \frac{dP_g^Z(t)}{dt} \text{ for any time } t \geq 0 \text{ a.e..} \tag{3}$$

Now, we give the definition of a possibly defective cdf. By Assumption 1, we naturally restrict ourselves to the absolute continuous cdf case.

Definition 3. A function $F : \mathbb{R}^+ \to [0,1]$ is a cdf if it satisfies the following properties. First, the function F is nondecreasing. Secondly, the function F is absolutely continuous, namely with pdf $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $f(t) = \frac{dF(t)}{dt}$ for any time $t \geq 0$ a.e. Thirdly, the function F satisfies $F(0) = 0$ and $\lim_{t\to\infty} F(t) = F_\infty \leq 1$ where $0 < F_\infty \leq 1$.

We introduce now the main problem of this paper. The IFPT problem determines a boundary function $g \in \mathcal{G}$ such that

$$f_g^Z(t) = f(t) \text{ for any time } t \geq 0 \text{ a.e.} \tag{4}$$

As explicit solutions are very limited, the literature related to the IFPT problem relies heavily on approximations. Based on Wang and Pötzelberger (1997) idea, an approximation to a continuous boundary by a piecewise linear boundary is given in Zucca and Sacerdote (2009). Their driving idea is to determine recursively the slope of the linear approximation on an interval by equating the probability of the FPT of the stochastic process Z to the approximation and the increment of the cdf on the interval. That approximation is based on the starting value of the boundary. Since the starting value of the boundary is unknown, it has to be guessed in practice. Moreover, they do not propose any asymptotics when the length of each interval of linear approximation goes to 0. These asymptotics are required to show that the approximated boundary function goes to the boundary function asymptotically.

In what follows, we consider a slight extension of the setting in Zucca and Sacerdote (2009) which has two main novelties. First, we approximate the starting value of the boundary. Second, we consider asymptotics where the length of each interval of linear approximation goes to 0 as the number of intervals $2^n \to \infty$. More specifically, we introduce the final time $t_f \in \mathbb{R}_*^+$. For any nonnegative integer $n \in \mathbb{N}$ and any nonnegative integer $m \in \{0, \dots, 2^n\}$, we consider a time discretization $t_m^n = m\Delta_n$. Here, $\Delta_n = 2^{-n} t_f$ is the length of each interval of linear approximation. We consider a nested time discretization as this will be required in the proof of Theorem 4.

We define the sequence of piecewise-linear approximation of the boundary function g^n recursively on the positive integer $m \in \{1, \dots, 2^n\}$ as

$$g^n(0) = \alpha_0^n, \tag{5}$$

$$g^n(u) = g^n(t_m^n) + \alpha_{m+1}^n (u - t_m^n) \text{ for any time } u \in (t_m^n, t_{m+1}^n]. \tag{6}$$

Here, we have that the coefficients $\alpha_0^n \in \mathbb{R}_*^+$ and $\alpha_m^n \in \mathbb{R}$ are defined implicitly for any positive integer $m \in \{1, \dots, 2^n\}$ by

$$P_{\alpha_0^n}^Z(\delta_n \Delta_n) = \int_0^{\delta_n \Delta_n} f(s) ds, \tag{7}$$

$$\mathbb{P}\big(T_{g^n}^Z \in [t_{m-1}^n, t_m^n]\big) = \int_{t_{m-1}^n}^{t_m^n} f(s) ds. \tag{8}$$

Equations (5), (6) and (8) correspond exactly to Equations (3.1)-(3.2) in Zucca and Sacerdote (2009). The novelty in this paper is Equation (7), in which we determine the approximation of the starting value of the boundary function. The idea is to equate the probability of the FPT of the stochastic process Z to a constant boundary value equal to the first coefficient α_0^n on an interval of length $\delta_n \Delta_n$. Here, the parameter $\delta_n \in (0, 1)$ corresponds to the proportion. The reason why we introduce the parameter δ_n is that there are numerical problems if we use $\delta_n = 1$. In practice, we recommend to use $\delta_n = 0.25$ which gives the best results. There are no theoretical problems, since our main results do not require any assumption on the parameter.

Our next result establishes that the sequence of coefficients α_m^n is well-defined. This is a slight extension of Remark 3.2 in Zucca and Sacerdote (2009), which also includes that the first coefficient α_0^n is well-defined.

Lemma 2. *We assume that Assumption 1 holds. For any nonnegative integer $n \in \mathbb{N}$, Equation (7) defines a unique coefficient $\alpha_0^n \in \mathbb{R}_*^+$ and Equation (8) defines a unique coefficient $\alpha_m^n \in \mathbb{R}$ for any nonnegative integer $m \in \{0, \dots, 2^n\}$.*

3. Main results

We first show that the starting value of the piecewise-linear approximation converges to the starting value of the boundary function. The elementary idea of the proof consists in observing that the boundary function can be bounded below and above by positive constants for a very small time interval. Then, we show that these constants converge to the boundary starting value as the interval length goes to 0. This is possible since the starting value of the boundary function is positive and the boundary function is continuous with our assumptions.

Proposition 3. *We assume that Assumption 1 holds. Then, the starting value of the piecewise-linear approximation converges to the starting value of the boundary function. Namely, we have $g^n(0) \to g(0)$ as the number of intervals $n \to \infty$.*

We now give our main result in the next theorem. This shows that a subsequence of the piecewise-linear approximation uniformly converges to the boundary function. The elementary idea of the proof consists in using Arzelà-Ascoli theorem on the compact space $[0, t_f]$. We first show that the coefficients α_m^n are uniformly bounded, which implies that the piecewise-linear approximation is uniformly bounded and uniformly equicontinuous. For that purpose, we assume that the boundary function is differentiable on the interval $[0, t_f]$ with uniformly dominated derivative.

Assumption 2. We assume that the boundary function g is differentiable on the interval $[0, t_f]$ with uniformly bounded derivatives, namely

$$\sup_{t \in [0, t_f]} \mid g'(t) \mid < \infty.$$

One limitation in the next result is that we only obtain the convergence of a subsequence, rather than a direct convergence. The reason is that we were not able to prove directly the convergence by extending the proving techniques used in Zucca and Sacerdote (2009).

Theorem 4. *We assume that Assumption 2 holds. Then, there exists a subsequence g^{n_k} of the piecewise-linear approximation g^n which converges uniformly to the boundary function g on the interval $[0, t_f]$. Namely, we have*

$$\sup_{t \in [0, t_f]} \mid g^{n_k}(t) - g(t) \mid \to 0$$

as the number of intervals $n \to \infty$.

4. Numerical study

In this section, we conduct a numerical study. We first show that the starting value of the piecewise-linear approximation is sensitive to the starting value of the boundary function and the starting value of the boundary function derivative. Then, we check the stability of the piecewise-linear approximation presented in Section 2 with the use of some examples where a closed form solution is available. We also consider another example where the solution is numerically evaluated. The results obtained in the numerical study indicate that the piecewise-linear approximation is adequate and relatively safe to use in practice. The results also illustrate that our asymptotics are adapted to obtain an adequate piecewise-linear approximation.

First, we report in Table 1 the normalized error for the starting value of the piecewise-linear approximation. We consider several linear boundaries with starting value equal to $0.5, 1, 2, 3, 4$ and slope equal to $0, 1, 2, 3, 4$. We choose the interval length $\Delta_n = 0.2$ and the proportion parameter $\delta_n = 0.25, 0.5$. The starting value of the piecewise-linear approximation is sensitive to the starting value of the linear boundary function and the slope of the linear boundary function. More specifically, the quality of the piecewise-linear approximation depends on the ratio of slope over starting value. When the slope is null, the absolute value of the normalized error is systematically below 0.10%. This can be explained by the fact that the boundary function is constant in that particular case. Overall, the absolute value of the normalized error is below 20.00% and with a positive bias for most instances. The case with smaller proportion parameter $\delta_n = 0.25$ reduces the normalized error by half compared to the case with larger proportion parameter $\delta_n = 0.5$ for most instances. This illustrates that our asymptotics based on the number of intervals $n \to \infty$ are adapted to obtain an adequate starting value of the piecewise-linear approximation.

Second, we define the Daniels boundary function and its pdf (see Daniels (1969)) for any time $t \geq 0$ as

$$\begin{aligned} g(t) &= \frac{\alpha}{2} - \frac{t}{\alpha} \log\Big(\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \gamma \exp\big(-\frac{\alpha^2}{t}\big)}\Big), \\ f_g(t) &= \frac{1}{\sqrt{2\pi t^3}}\Big(\exp\big(-\frac{g(t)^2}{2t}\big) - \frac{2}{\beta}\exp\big(\frac{(g(t)-\alpha)^2}{2t}\big)\Big). \end{aligned}$$

Here, we have that the parameters satisfy $\alpha > 0$, $\beta \geq 0$ and $\gamma > \beta/4$. We also define the oscillating boundary function for any time $t \geq 0$ as

$$g(t) = \alpha + \beta\cos(\gamma t).$$

Here, we have that the parameter α is the starting value of the oscillating boundary function and satisfies $\alpha > 0$. Moreover, the parameter β corresponds to the amplitude and satisfies $\beta > 0$. Finally the parameter γ corresponds to the period of an oscillation and satisfies $\gamma > 0$. Since there is no explicit formula for the pdf f_g, we evaluate numerically its value by Buonocore, Nobile and Ricciardi (1987).

TABLE 1
Normalized error for the starting value of the piecewise-linear approximation. We consider several linear boundaries with starting value equal to $0.5, 1, 2, 3, 4$ *and slope equal to* $0, 1, 2, 3, 4$. *We choose the interval length* $\Delta_n = 0.2$ *and the proportion parameter* $\delta_n = 0.25, 0.5$.

Boundary starting		$\delta_n = 0.25$				
Value	Slope	0	1	2	3	4
0.5		-0.10%	8.60%	17.38%	26.20%	35.04%
1		-0.10%	4.69%	9.49%	14.29%	19.10%
2		-0.10%	2.37%	4.84%	7.32%	9.79%
3		-0.10%	1.56%	3.22%	4.88%	6.54%
4		-0.10%	2.43%	4.92%	7.41%	9.89%
Boundary starting		$\delta_n = 0.5$				
Value	Slope	0	1	2	3	4
0.5		-0.06%	15.90%	32.14%	48.52%	64.92%
1		-0.05%	9.18%	18.47%	27.78%	37.10%
2		-0.05%	4.83%	9.73%	14.62%	19.52%
3		-0.05%	3.25%	6.55%	9.85%	13.15%
4		-0.05%	2.43%	4.92%	7.41%	9.89%

TABLE 2
Mean squared error of the piecewise-linear approximations for the three boundary functions. We choose the interval length $\Delta_n = 0.2$ *and the proportion parameter* $\delta_n = 0.25, 0.5, 1$.

Daniels boundary with $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$	
Proportion parameter δ_n	Mean squared error
0.25	1.66×10^{-5}
0.5	5.96×10^{-5}
1	5.89×10^{-5}
Daniels boundary with $\alpha = 1$, $\beta = 0.5$, $\gamma = 0.5$	
Proportion parameter δ_n	Mean squared error
0.25	1.82×10^{-4}
0.5	3.56×10^{-4}
1	1.07×10^{-3}
Oscillating boundary with $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 8$	
Proportion parameter δ_n	Mean squared error
0.25	7.09×10^{-2}
0.5	6.96×10^{-2}
1	7.21×10^{-2}

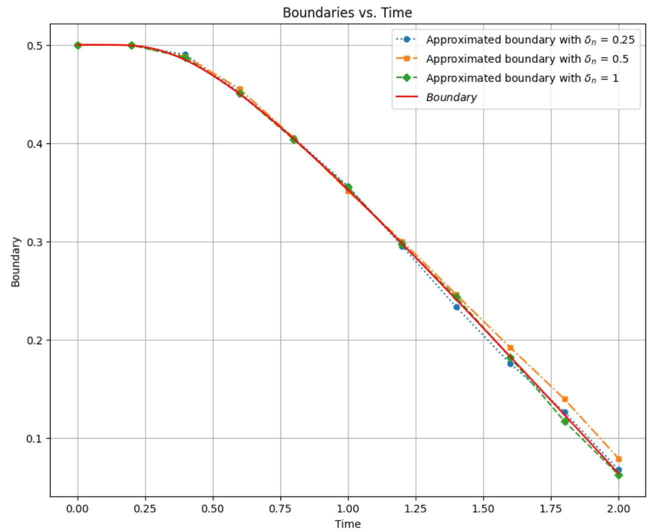

FIG 1. *Daniels boundary function with parameters* $\alpha = 1$, $\beta = 1$ *and* $\gamma = 0.5$ *compared with the piecewise-linear approximation as a function of time* $t \in [0, 2]$. *We choose the interval length* $\Delta_n = 0.2$ *and the proportion parameter* $\delta_n = 0.25, 0.5, 1$. *with* $\Delta_n = 0.2$ *and* $\delta_n = 0.25, 0.5, 1$.

Figure 1 plots the Daniels boundary function with parameters $\alpha = 1$, $\beta = 1$ and $\gamma = 0.5$ compared with the piecewise-linear approximation as a function of time $t \in [0, 2]$. We choose the interval length $\Delta_n = 0.2$ and the proportion parameter $\delta_n = 0.25, 0.5, 1$. with $\Delta_n = 0.2$ and $\delta_n = 0.25, 0.5, 1$. As the starting value of the boundary function derivative is 0, the starting value of the three piecewise-linear approximations are adequate.

Figure 2 plots the Daniels boundary function with parameters $\alpha = 1$, $\beta = 0.5$ and $\gamma = 0.5$ compared with the piecewise-linear approximation as a function of time $t \in [0, 2]$. We choose the interval length $\Delta_n = 0.2$ and the proportion parameter $\delta_n = 0.25, 0.5, 1$. The starting value of the boundary function derivative is around unity. The piecewise-linear approximation is adequate when $\delta_n = 0.25$, but not as accurate when $\delta_n = 0.5$ and a bit off when $\delta_n = 1$. As the time increases, the piecewise-linear approximation tends to oscillate around the boundary. This is due to the fact that the approximation overcompensates by its own definition (8). This also illustrates that our asymptotics based on the number of intervals $n \to \infty$ are adapted to obtain an adequate starting value of the piecewise-linear approximation.

Figure 3 plots the oscillating boundary function with parameters $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 8$ compared with the piecewise-linear approximation as a function of time $t \in [0, 2]$. We choose the interval length $\Delta_n = 0.2$ and the proportion parameter $\delta_n = 0.25, 0.5, 1$. This is a more complicated case as the oscillating boundary function is not monotone. As for Figure 2, the approximation is adequate when $\delta_n = 0.25$, but not as accurate when $\delta_n = 0.5$ and a bit off

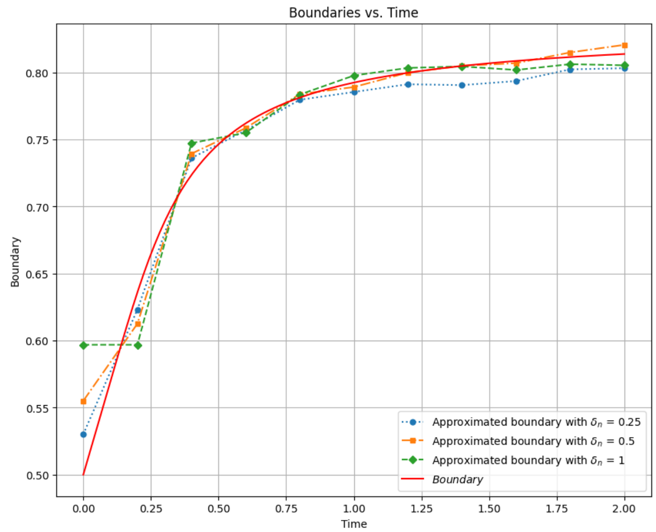

FIG 2. *Daniels boundary function with parameters $\alpha = 1$, $\beta = 0.5$ and $\gamma = 0.5$ compared with the piecewise-linear approximation as a function of time $t \in [0, 2]$. We choose the interval length $\Delta_n = 0.2$ and the proportion parameter $\delta_n = 0.25, 0.5, 1$.*

when $\delta_n = 1$. Moreover, all the piecewise-linear approximations are slightly off at times where the monotonicity of the oscillating boundary function changes. This documents the limitation of the method of approximation. All these results are confirmed by Table 2, which reports the mean squared error of the piecewise-linear approximations for the three boundary functions.

5. Proofs from the preliminary results

In this section, we give the proofs from the preliminary results, which are elementary. However, some ideas and notations are used for the proofs of the main results given in Section 6. Thus, the reader interested in the proofs of the main results should use this section as a reference.

We start with the proof of Lemma 1, which slightly extends the arguments from the proof of Lemma 3.3 in Strassen (1967).

Proof of Lemma 1. By Assumption 1, we have that the boundary function g is absolutely continuous on the set of nonnegative real numbers $\mathbb{R}^+$. Thus, the boundary function g admits a derivative a.e. on the set of nonnegative real numbers $\mathbb{R}^+$. Namely, there exists a Lebesgue-negligible set $\mathcal{N} \subset \mathbb{R}^+$ such that the boundary function g admits a derivative for any time $t \in \mathbb{R}^+ - \mathcal{N}$.

To show that the cdf P_g^W is absolutely continuous on the set of nonnegative real numbers $\mathbb{R}^+$, it is sufficient to show that the cdf P_g^W admits a derivative for any time $t \in \mathbb{R}^+ - \mathcal{N}$. This is due to the fact that $\mathcal{N}$ is a Lebesgue-negligible

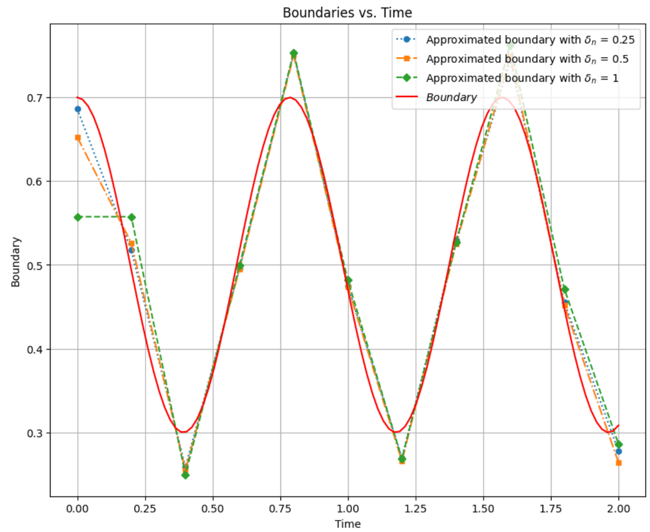

FIG 3. *Oscillating boundary function with parameters* $\alpha = 0.5$, $\beta = 0.2$ *and* $\gamma = 8$ *compared with the piecewise-linear approximation as a function of time* $t \in [0, 2]$. *We choose the interval length* $\Delta_n = 0.2$ *and the proportion parameter* $\delta_n = 0.25, 0.5, 1$.

set. By the definition of absolute continuity, we have that the set $\mathcal{N}$ is countable on any compact space of $\mathbb{R}^+$. Indeed, the boundary function g does not admit a derivative in the neighborhood of the time s if $s \in \mathbb{R}^+$ is an accumulation point of $\mathcal{N}$. Thus, the boundary function g is not absolutely continuous. Then, we have that the set $\mathcal{N}$ is countable on any compact space of $\mathbb{R}^+$.

To show that the cdf P_g^W is absolutely continuous on $\mathbb{R}^+$, it is then sufficient to show that the cdf P_g^W admits a derivative on any open interval (u, v) where the times $u \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ satisfy $u < v$ and $(u, v) \cap \mathcal{N} = \emptyset$. We can show this statement by extending the arguments from the proof of Lemma 3.3 in Strassen (1967) along with the assumption that $g(0) > 0$ from Definition 1.

Finally, the reflected Wiener process case follows since the FPT of a reflected Wiener process to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal. $\square$

In the next definition, we introduce the transition pdf of a stochastic process constrained by an absorbing boundary.

Definition 4. We define the transition pdf of the stochastic process Z at the time $t \in \mathbb{R}^+$ constrained by the absorbing boundary function g over the interval $[s, t]$ given that $Z_s = y$ as $p_g^Z(t, x \mid s, y)$. The transition pdf satisfies for any

$x < g(t)$, $t > s \geq 0$ and $y < g(s)$ that

$$p_g^Z(t, x \mid s, y) = \frac{\partial}{\partial x}\mathbb{P}(Z_t < x, \mathrm{T}_g^Z > t \mid Z_s = y). \tag{9}$$

In the following lemma, we give the pdf and the transition pdf for the FPT of a Wiener process to a linear boundary. This is a consequence to Doob (1949) (Equation (4.2), p. 397), Malmquist (1954) (p. 526) and Durbin (1971) (Lemma 1).

Lemma 5. *We assume that the boundary function is linear*

$$g(t) = \alpha_1 t + \alpha_0 \text{ for any time } t \geq 0.$$

Here, $t_0 \geq 0$, $\alpha_0 \in \mathbb{R}_*^+$, $\alpha_1 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ satisfy $g(t_0) > x_0$. Then, we have that the pdf for the FPT of a Wiener process is equal to

$$f_g^W(t \mid t_0, x_0) \quad = \quad \frac{\alpha_0 - x_0}{\sqrt{2\pi(t-t_0)^3}} \exp\Big(-\frac{(\alpha_0 + \alpha_1(t-t_0) - x_0)^2}{2(t-t_0)}\Big). \tag{10}$$

Moreover, the transition pdf is equal to

$$p_g^W(t_1, x_1 \mid t_0, x_0) \quad = \quad \Big(1 - \exp\Big(\frac{-2(g(t_1) - x_1)(g(t_0) - x_0)}{t_1 - t_0}\Big)\Big)$$
$$\times \frac{\exp(-\frac{(x_1 - x_0)^2}{2(t_1 - t_0)})}{\sqrt{2\pi(t_1 - t_0)}}. \tag{11}$$

Proof of Lemma 5. Equation (10) is obtained in Doob (1949) (Equation (4.2), p. 397) or Malmquist (1954) (p. 526). Equation (11) follows from Durbin (1971) (Lemma 1). □

In the following lemma, we give the pdf and the transition pdf for the FPT of a Wiener process to a continuous piecewise linear boundary. Equation (13) is already available in Wang and Pötzelberger (1997) and Zucca and Sacerdote (2009) (Section 2.1.3, pp. 1323-1324).

Lemma 6. *We assume that the boundary function is piecewise linear*

$$g(t) = \alpha_i t + \beta_i, \text{ for any time } t \in [t_{i-1}, t_i].$$

Here, we have that $t_i = i\Delta + t_0$, where $t_0 \geq 0$, $\Delta > 0$ and the coefficients $\alpha_i, \beta_i \in \mathbb{R}$ satisfy $\alpha_{i+1} + \beta_{i+1} t_i = \alpha_i + \beta_i t_i$. Thus, the boundary is continuous. Then, we can express the transition pdf as

$$p_g^W(t_1, x_1, \ldots, t_n, x_n \mid t_0, x_0) \quad = \quad \prod_{i=1}^{n} p_g^W(t_i, x_i \mid t_{i-1}, x_{i-1}). \tag{12}$$

Here, we have that $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $x_i \leq g(t_i)$ for any nonnegative integer $i = 1, \ldots, n$ and $x_0 < g(t_0)$ where $t_0 < t_1 < t_2 < \ldots < t_n$. Moreover, we *can reexpress the transition pdf with the following explicit expression*

$$\begin{aligned} p_g^W(t_1, x_1, \dots, t_n, x_n \mid t_0, x_0) &= \prod_{i=1}^{n} \Big(1 - \\ &\quad \exp\big(\frac{-2(g(t_i) - x_i)(g(t_{i-1}) - x_{i-1})}{t_i - t_{i-1}}\big)\Big) \\ &\quad \times \frac{\exp\Big(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\Big)}{\sqrt{2\pi(t_i - t_{i-1})}}. \end{aligned} \tag{13}$$

Finally, we can deduce for any Borel set $C_i \subset (-\infty, g(t_i)]$ *with* $i = 1, \dots, n$ *that*

$$\begin{aligned} &\mathbb{P}(W_{t_1} \in C_1, \dots, W_{t_n} \in C_n, \mathrm{T}_g^W > t_n \mid W_{t_0} = x_0) \qquad (14) \\ &= \int_{C_1} \dots \int_{C_n} p_g^W(t_1, x_1, \dots, t_n, x_n \mid t_0, x_0) dx_1 \dots dx_n. \end{aligned}$$

Proof of Lemma 6. Equation (12) is obtained by Definition (9) and follows by induction with conditional probability formula. Then, Equation (13) can be deduced by plugging Equation (11) into Equation (12). Finally, Equation (14) is a direct consequence of Equation (13). □

We define ϕ as the standard Gaussian cdf. In the following lemma, we give the pdf for the FPT of a reflected Wiener process to a linear boundary. This is based on the explicit solution from Anderson (1960) (Theorem 5.1, p. 191) for the FPT to an upper linear boundary and a lower linear boundary. This is due to the fact that the FPT of a reflected Wiener process to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal. Although we can deduce the transition pdf and transition pdf for the piecewise linear boundary with the same arguments as in the proofs of Lemma 5 and Lemma 6, we do not report them in the following of this paper.

Lemma 7. *We assume that the boundary function is linear*

$$g(t) = \alpha_1 t + \alpha_0 \text{ for any time } t \geq 0.$$

Here, we have that $t_0 \geq 0$ *is the starting time,* $\alpha_0 \in \mathbb{R}_*^+$ *is the intercept,* $\alpha_1 \in \mathbb{R}$ *is the trend and the starting value* $x_0 \in \mathbb{R}$ *satisfy* $g(t_0) > x_0$*. Then, we have* *that the pdf for the FPT of a reflected Wiener process is equal to*

$$f_g^{|W|}(t_0 \mid t_0, x_0) = 0, \tag{15}$$

$$\begin{aligned} f_g^{|W|}(t \mid t_0, x_0) = \; & \frac{2}{(t-t_0)^{3/2}} \phi\Big(\frac{\alpha_1(t-t_0)+\alpha_0-x_0}{\sqrt{t-t_0}}\Big) \\ & \times \sum_{r=0}^{\infty} \Big\{ (4r+1)(\alpha_0-x_0) \\ & \times \exp\Big(\frac{-(8r(r+1)(\alpha_0-x_0))(\alpha_1(t-t_0)+\alpha_0-x_0)}{t-t_0}\Big) \\ & -(4r+2)(\alpha_0-x_0)\exp\Big(\\ & \frac{-(4(r+1)(2r+1)(\alpha_0-x_0)(\alpha_1(t-t_0)+\alpha_0-x_0)}{t-t_0}\Big)\Big\} \\ & \textit{for any time } t > t_0. \end{aligned} \tag{16}$$

Proof of Lemma 7. The proof follows from the fact that we have the FPT of a reflected Wiener to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal.

More specifically, we first consider the FPT of a Wiener process to an upper linear boundary and a lower linear boundary. We first assume that the boundary is upper linear and lower linear. Namely, we assume that

$$g(t) = (\gamma_2 + \delta_2 t, \gamma_1 + \delta_1 t).$$

Here, we have that $\gamma_1 > 0$, $\gamma_2 < 0$, $\delta_1 \geq \delta_2$ and we do not have that $\delta_1 = \delta_2 = 0$. By Anderson (1960) (Theorem 5.1, p. 191), we have that the pdf of the FPT is equal for any time $t > t_0$ to

$$
\begin{aligned}
f_g^W(t_0) &= 0, && (17)\\
f_g^W(t) &= \frac{1}{(t-t_0)^{3/2}}\Bigg[\phi\Big(\frac{\delta_1(t-t_0)+\gamma_1}{\sqrt{t-t_0}}\Big)\sum_{r=0}^{\infty}\Big\{\big((2r+1)\gamma_1-2r\gamma_2\big) && (18)\\
&\times\exp\Big(\frac{-2r(r\gamma_1-(r+1)\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\\
&-\big(2(r+1)\gamma_1-2r\gamma_2\big)\exp\Big(\\
&\frac{-2(r+1)((r+1)\gamma_1-r\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\Big\}\\
&+\phi\Big(\frac{\delta_2(t-t_0)+\gamma_2}{\sqrt{t-t_0}}\Big)\sum_{r=0}^{\infty}\Big\{\big(2r\gamma_1-(2r+1)\gamma_2\big)\exp\Big(\\
&\frac{-2(r+1)((r+1)\gamma_1-r\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2(t-t_0)+\gamma_2))}{t-t_0}\Big)\\
&-\big(2(r+1)\gamma_1-2r\gamma_2\big)\\
&\exp\Big(\frac{-2r(r\gamma_1-(r+1)\gamma_2)(\delta_1(t-t_0)+\gamma_1-(\delta_2 t+\gamma_2))}{t-t_0}\Big)\Big\}\Bigg],
\end{aligned}
$$

Now, we assume that the boundaries are symmetric. Namely, we assume that $g(t) = (-\alpha_1 t - \alpha_0, \alpha_1 t + \alpha_0)$ where $\alpha_1 \in \mathbb{R}$ and $\alpha_0 \in \mathbb{R}_*^+$. From Equations (17) and (18), we can deduce that

$$
\begin{aligned}
f_g^W(t_0) &= 0, && (19)\\
f_g^W(t) &= \frac{2}{(t-t_0)^{3/2}}\phi\Big(\frac{\alpha_1(t-t_0)+\alpha_0}{\sqrt{t-t_0}}\Big) && (20)\\
&\sum_{r=0}^{\infty}\Big\{(4r+1)\alpha_0\exp\Big(\frac{-(8r(r+1)\alpha_0)(\alpha_1 t-t_0+\alpha_0)}{t-t_0}\Big)\\
&-(4r+2)\alpha_0\exp\Big(\frac{-(4(r+1)(2r+1)\alpha_0(\alpha_1(t-t_0)+\alpha_0)}{t-t_0}\Big\}\\
&\text{for any time } t > t_0.
\end{aligned}
$$

We have the FPT of a reflected Wiener to a linear boundary is equal to the FPT of a Wiener process to a symmetric upper linear boundary and lower linear boundary when the boundary from the reflected Wiener process and the upper boundary are equal. From the previous sentence with Equations (19) and (20), we can deduce Equations (15) and (16).

□

The next lemma gives the transition pdf for a FPT of a Wiener process W at time t_{m+1}^n constrained by the absorbing boundary function g^n over the interval $[t_m^n, t_{m+1}^n]$ given that $W_{t_m} = x_m$.

Lemma 8. *For any positive integer $n \in \mathbb{N}_*$ and any nonnegative integer $m \in \{0, \dots, 2^n - 1\}$, we have*

$$p_{g^n}^W\big(t_{m+1}^n, x_{m+1} \mid t_m^n, x_m\big) = \Big(1 - \exp\Big(\frac{-2(g^n(t_{m+1}^n) - x_{m+1})(g^n(t_m^n) - x_m)}{\Delta_n}\Big)\Big) \times \frac{\exp\big(-\frac{(x_{m+1}-x_m)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}}. \tag{21}$$

Proof of Lemma 8. Equation (21) can be obtained directly from Equation (11) in Lemma 5. □

We define the probability of the FPT to a constant boundary function equal to $\alpha \in \mathbb{R}_*^+$ on the first interval $[0, \delta_n\Delta_n]$ as $G_0^n : \mathbb{R}_*^+ \to \mathbb{R}$ which satisfies

$$G_0^n(\alpha) = 1 - \int_{-\infty}^{\alpha} \Big(1 - \exp\Big(\frac{-2(\alpha - x_1)\alpha}{\delta_n\Delta_n}\Big)\Big) \frac{\exp\big(-\frac{x_1^2}{\delta_n\Delta_n}\big)}{\sqrt{\pi\delta_n\Delta_n}} dx_1. \tag{22}$$

The next lemma gives a more explicit form to the starting coefficient α_0^n.

Lemma 9. *For any positive integer $n \in \mathbb{N}$, Equation (7) can be reexpressed as*

$$G_0^n(\alpha_0^n) - \int_0^{\delta_n\Delta_n} f(s)ds = 0. \tag{23}$$

Proof of Lemma 9. We have that

$$\begin{aligned} P_{\alpha_0^n}^W(\delta_n\Delta_n) &= \mathbb{P}\big(T_{\alpha_0^n}^W \in [0, \delta_n\Delta_n]\big) \\ &= \mathbb{P}\big((T_{\alpha_0^n}^W > \delta_n\Delta_n)^C\big) \\ &= 1 - \mathbb{P}\big(T_{\alpha_0^n}^W > \delta_n\Delta_n\big) \\ &= 1 - \mathbb{P}\big(W_{\delta_n\Delta_n} \in (-\infty, \alpha_0^n], T_{\alpha_0^n}^W > \delta_n\Delta_n\big) \\ &= 1 - \int_{-\infty}^{\alpha_0^n} p_{\alpha_0^n}^W(\delta_n\Delta_n, x_1 \mid 0, 0)dx_1 \\ &= 1 - \int_{-\infty}^{\alpha_0^n} \Big(1 - \exp\Big(\frac{-2(\alpha_0^n - x_1)\alpha_0^n}{\delta_n\Delta_n}\Big)\Big) \\ &\quad \times \frac{\exp\big(-\frac{x_1^2}{\delta_n\Delta_n}\big)}{\sqrt{\pi\delta_n\Delta_n}} dx_1. \end{aligned} \tag{24}$$

Here, we use Equation (2) in the first equality and the fact that $T_{\alpha_0^n}^W \geq 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the second equality. We also use elementary probability facts in the third equality and the fact that $T_{\alpha_0^n}^W \subset \{W_{\delta_n\Delta_n} \in (-\infty, \alpha_0^n]\}$ in the fourth equality. Moreover, we use Equation (14) from Lemma 6 in the fifth equality and Equation (21) from Lemma 8 in the sixth equality. Finally, we can deduce Equation (23) by plugging Equation (7) into Equation (24). □

We define the probability of the FPT to a linear boundary function started from α_0^n with trend $\alpha \in \mathbb{R}$ on the interval $[0, t_1^n]$ as $G_1^n : \mathbb{R} \to \mathbb{R}$ which satisfies

$$G_1^n(\alpha) = 1 - \int_{-\infty}^{\alpha_0^n + \alpha\Delta_n} \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \alpha\Delta_n - x_1)\alpha_0^n}{\Delta_n}\Big)\Big) \frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi\Delta_n}} dx_1. \tag{25}$$

In the next lemma, we give a more explicit form to the coefficient α_1^n based on the known coefficient value α_0^n. The idea of the proof is similar to the idea of the proof from Lemma 9.

Lemma 10. *For any nonnegative integer $n \in \mathbb{N}$ and $m = 1$, Equation (8) can be reexpressed as*

$$G_1^n(\alpha_1^n) - \int_0^{t_1^n} f(s)ds = 0. \tag{26}$$

Proof of Lemma 10. We have that

$$\begin{aligned}
\mathbb{P}\big(T_{g^n}^W \in [0, t_1^n]\big) &= \mathbb{P}\big((T_{g^n}^W > t_1^n)^C\big) \\
&= 1 - \mathbb{P}\big(T_{g^n}^W > t_1^n\big) \\
&= 1 - \mathbb{P}\big(W_{t_1^n} \in (-\infty, g^n(t_1^n)], T_{g^n}^W > t_1^n\big) \\
&= 1 - \int_{-\infty}^{g^n(t_1^n)} p_{g^n}^W(t_1^n, x_1 \mid 0, 0) dx_1 \\
&= 1 - \int_{-\infty}^{g^n(t_1^n)} \Big(1 - \exp\Big(\frac{-2(g^n(t_1^n) - x_1)g^n(0)}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp\big(-\frac{x_1^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} dx_1 \\
&= 1 - \int_{-\infty}^{\alpha_0^n + \alpha_1^n\Delta_n} \Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \alpha_1^n\Delta_n - x_1)\alpha_0^n}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp\big(-\frac{x_1^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} dx_1.
\end{aligned} \tag{27}$$

Here, we use the fact that $T_{g^n}^W \geq 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the first equality and elementary probability facts in the second equality. We also use the fact that $T_{g^n}^W \subset \{W_{t_1^n} \in (-\infty, g^n(t_1^n)]\}$ in the third equality and Equation (14) from Lemma 6 in the fourth equality. Moreover, we use Equation (21) from Lemma 8 in the fifth equality and Equations (5)-(6) in the sixth equality. Finally, we can deduce Equation (26) by plugging Equation (8) into Equation (27). □

We define now the probability of the FPT to a continuous piecewise linear boundary g^n on the first interval $[0, t_1^n]$ and with trend $\alpha \in \mathbb{R}$ on the second interval $[t_1^n, t_2^n]$ as $G_2^n : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\begin{aligned} G_2^n(\alpha) &= 1 - G_1^n(\alpha_1^n) - \int_{-\infty}^{\alpha_0^n+\alpha_1^n\Delta_n} \int_{-\infty}^{\alpha_0^n+(\alpha_1^n+\alpha)\Delta_n} \\ &\Big(1 - \exp\Big(\frac{-2(\alpha_0^n + (\alpha_1^n+\alpha)\Delta_n - x_2)(\alpha_0^n + \alpha_1^n\Delta_n - x_1)}{\Delta_n}\Big)\Big) \\ &\frac{\exp(-\frac{(x_2-x_1)^2}{\Delta_n})}{\sqrt{\pi\Delta_n}} \\ &\times\Big(1 - \exp\Big(\frac{-2(\alpha_0^n + \alpha_1^n\Delta_n - x_1)\alpha_0^n}{\Delta_n}\Big)\Big)\frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi\Delta_n}} dx_1 dx_2. \end{aligned} \tag{28}$$

In the next lemma, we give a more explicit form to α_2^n based on the known values α_1^n and α_0^n. The idea of the proof is similar to the idea of the proof from Lemma 9.

Lemma 11. *For any nonnegative integer $n \in \mathbb{N}$ and $m = 2$, Equation (8) can be reexpressed as*

$$G_2^n(\alpha_2^n) - \int_{t_1^n}^{t_2^n} f(s)ds = 0.$$

Proof of Lemma 11. We have that

$$\begin{aligned} \mathbb{P}\big(T_{g^n}^W \in [t_1^n, t_2^n]\big) &= \mathbb{P}\left((\mathrm{T}_{g^n}^W < t_1^n, \mathrm{T}_{g^n}^W > t_2^n)^C\right) \\ &= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_1^n, \mathrm{T}_{g^n}^W > t_2^n\right) \\ &= 1 - \mathbb{P}\left(\mathrm{T}_{g^n}^W < t_1^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\ &= 1 - \mathbb{P}\left(0 \le \mathrm{T}_{g^n}^W < t_1^n\right) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\ &= 1 - \int_0^{t_1^n} f(s)ds - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right) \\ &= 1 - G_1^n(\alpha_1^n) - \mathbb{P}\left(\mathrm{T}_{g^n}^W > t_2^n\right). \end{aligned} \tag{29}$$

Here, we use elementary probability facts in the first and second equalities. We also use the fact that $\{\mathrm{T}_{g^n}^W < \Delta_n\}$ and $\{\mathrm{T}_{g^n}^W > 2\Delta_n\}$ are disjoint events in the third equality and the fact that $T_{g^n}^W \ge 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the fourth equality. Moreover, we use Equation (8) in the fifth equality and Lemma 10 in the sixth equality.

Also, we have that

$$
\begin{aligned}
\mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_2\right) &= \mathbb{P}\big(W_{t^n_1} \in (-\infty, g^n(t^n_1)], W_{t^n_2} \in (-\infty, g^n(t^n_2)], \mathrm{T}^W_{g^n} > t^n_2\big) \\
&= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} p^W_{g^n}(t^n_1, x_1, t^n_2, x_2 \mid 0, 0) dx_1 dx_2 \\
&= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} p^W_{g^n}(t^n_2, x_2 \mid t^n_1, x_1) p^W_{g^n}(t^n_1, x_1 \mid 0, 0) dx_1 dx_2 \\
&= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} \\
&\quad \Big(1 - \exp\Big(\frac{-2(g^n(t^n_2) - x_2)(g^n(t^n_1) - x_1)}{\Delta_n}\Big)\Big) \frac{\exp\big(-\frac{(x_2 - x_1)^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}} \\
&\quad \times (1 - \exp(\frac{-2(g^n(t^n_1) - x_1) g^n(t^n_0)}{\Delta_n})) \frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} dx_1 dx_2.
\end{aligned}
$$

Here, we use the fact that

$$
\{\mathrm{T}^W_{g^n} > t^n_2\} \subset \{W_{t^n_1} \in (-\infty, g^n(t^n_1)], W_{t^n_2} \in (-\infty, g^n(t^n_2)]\}
$$

in the first equality. We also use Equation (14) in the second equality, Equation (12) in the third equality and Equation (21) from Lemma 8 in the fourth equality.

Then, we have that

$$
\begin{aligned}
\mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_2\right) &= \int_{-\infty}^{\alpha^n_0 + \alpha^n_1 \Delta_n} \int_{-\infty}^{\alpha^n_0 + (\alpha^n_1 + \alpha^n_2)\Delta_n} \\
&\quad \Big(1 - \exp\Big(\frac{-2(\alpha^n_0 + (\alpha^n_1 + \alpha^n_2)\Delta_n - x_2)(\alpha^n_0 + \alpha^n_1 \Delta_n - x_1)}{\Delta_n}\Big)\Big) \\
&\quad \frac{\exp(-\frac{(x_2 - x_1)^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} \\
&\quad \times \Big(1 - \exp\Big(\frac{-2(\alpha^n_0 + \alpha^n_1 \Delta_n - x_1)\alpha^n_0}{\Delta_n}\Big)\Big) \frac{\exp(-\frac{x_1^2}{\Delta_n})}{\sqrt{\pi \Delta_n}} dx_1 dx_2.
\end{aligned} \tag{30}
$$

Here, we use Equations (5)-(6). Finally, we can deduce Equation (29) by plugging Equation (30) and Equation (8) into Equation (29). □

For any positive integer $m \in \{3, \dots, 2^n - 1\}$, we define $x_0 = 0$ and the probability of the FPT to a continuous piecewise linear boundary g^n on the interval $[0, t^n_{m-1}]$ and with trend $\alpha \in \mathbb{R}$ on the interval $[t^n_{m-1}, t^n_m]$ as $G^n_m : \mathbb{R} \to \mathbb{R}$

which satisfies

$$\begin{aligned} G^n_m(\alpha) &= 1 - \sum_{k=1}^{m-1} G^n_k(\alpha^n_k) - \int_{-\infty}^{\alpha^n_0+\alpha^n_1\Delta_n} \int_{-\infty}^{\alpha^n_0+(\alpha^n_1+\alpha^n_2)\Delta_n} \dots \\ & \int_{-\infty}^{\alpha^n_0+(\sum_{k=1}^m \alpha^n_k+\alpha)\Delta_n} \Big(1 - \exp\Big(\\ & \frac{-2(\alpha^n_0 + (\sum_{i=1}^m \alpha^n_i + \alpha)\Delta_n - x_{m+1})(\alpha^n_0 + \sum_{i=1}^m \alpha^n_i \Delta_n - x_m)}{\Delta_n}\Big)\Big) \\ & \times \frac{\exp\big(-\frac{(x_{m+1}-x_m)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} \prod_{k=0}^{m-1} \Big(1 - \\ & \exp\big(\frac{-2(\alpha^n_0 + \sum_{i=1}^{k+1} \alpha^n_i \Delta_n - x_{k+1})(\alpha^n_0 + \sum_{i=1}^{k} \alpha^n_i \Delta_n - x_k)}{\Delta_n}\Big)\Big) \\ & \times \frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi\Delta_n}} dx_1 dx_2 \dots dx_m. \end{aligned} \tag{31}$$

In the lemma that follows, we give a more explicit form to the unknown coefficient α^n_m based on known coefficient values $(\alpha^n_k)_{k=0,\dots,m-1}$. The idea of the proof is similar to the idea of the proof from Lemma 9.

Lemma 12. *For any nonnegative integer $n \in \mathbb{N}$ and any positive integer $m \in \{3, \dots, 2^n - 1\}$, Equation (8) can be reexpressed as*

$$G^n_m(\alpha^n_m) - \int_{t^n_{m-1}}^{t^n_m} f(s)ds = 0. \tag{32}$$

Proof of Lemma 12. We have that

$$\begin{aligned} \mathbb{P}\big(T^W_{g^n} \in [t^n_{m-1}, t^n_m]\big) &= \mathbb{P}\left((\mathrm{T}^W_{g^n} < t^n_{m-1}, \mathrm{T}^W_{g^n} > t^n_m)^C\right) \\ &= 1 - \mathbb{P}\left(\mathrm{T}^W_{g^n} < t^n_{m-1}, \mathrm{T}^W_{g^n} > t^n_m\right) \\ &= 1 - \mathbb{P}\left(\mathrm{T}^W_{g^n} < t^n_{m-1}\right) - \mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right) \\ &= 1 - \mathbb{P}\left(0 \le \mathrm{T}^W_{g^n} < t^n_{m-1}\right) - \mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right) \\ &= 1 - \int_0^{t^n_{m-1}} f(s)ds - \mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right) \\ &= 1 - \sum_{k=1}^{m-1} G^n_k(\alpha^n_k) - \mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right). \end{aligned} \tag{33}$$

Here, we use elementary probability facts in the first and second equalities. We also use the fact that $\{\mathrm{T}^W_{g^n} < t^n_{m-1}\}$ and $\{\mathrm{T}^W_{g^n} > t^n_m\}$ are disjoint events in the third equality and the fact that $T^W_{g^n} \ge 0$ a.s. by Definition 2 along with the completeness of the filtration $\mathbf{F}$ in the fourth equality. Moreover, we use Equation (8) in the fifth equality and Lemmas 10 and 11 in the sixth equality.

Also, we have that

$$\begin{aligned}\mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right) &= \mathbb{P}\Big(W_{t^n_1} \in (-\infty, g^n(t^n_1)], W_{t^n_2} \in (-\infty, g^n(t^n_2)], \dots, \\ &\quad W_{t^n_m} \in (-\infty, g^n(t^n_m)], \mathrm{T}^W_{g^n} > t^n_m\Big) \\ &= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} \dots \int_{-\infty}^{g^n(t^n_m)} p^W_{g^n}(t^n_1, x_1, t^n_2, x_2, \dots, \\ &\quad t^n_m, x_m \mid 0, 0) dx_1 dx_2 \dots dx_m \\ &= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} \dots \int_{-\infty}^{g^n(t^n_m)} \prod_{k=0}^{m-1} p^W_{g^n}(t^n_{k+1}, x_{k+1} \mid t^n_k, x_k) \\ &\quad dx_1 dx_2 \dots dx_m.\end{aligned}$$

Here, we use the fact that

$$\{\mathrm{T}^W_{g^n} > t^n_m\} \subset \{W_{t^n_1} \in (-\infty, g^n(t^n_1)], W_{t^n_2} \in (-\infty, g^n(t^n_2)], \dots, W_{t^n_m} \in (-\infty, g^n(t^n_m)]\}$$

in the first equality. We also use Equation (14) in the second equality and Equation (12) in the third equality.

Then, we have that

$$\begin{aligned}\mathbb{P}\left(\mathrm{T}^W_{g^n} > t^n_m\right) &= \int_{-\infty}^{g^n(t^n_1)} \int_{-\infty}^{g^n(t^n_2)} \dots \int_{-\infty}^{g^n(t^n_m)} \prod_{k=0}^{m-1} \Big(1 - \\ &\quad \exp\Big(\frac{-2(g^n(t^n_{k+1}) - x_{k+1})(g^n(t^n_k) - x_k)}{\Delta_n}\Big)\Big) \\ &\quad \times \frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}} dx_1 dx_2 \dots dx_m, \\ &= \int_{-\infty}^{\alpha^n_0 + \alpha^n_1 \Delta_n} \int_{-\infty}^{\alpha^n_0 + (\alpha^n_1 + \alpha^n_2)\Delta_n} \dots \int_{-\infty}^{\alpha^n_0 + \sum_{k=1}^m \alpha^n_k \Delta_n} \prod_{k=0}^{m-1} \\ &\quad \Big(1 - \exp\Big(\\ &\quad \frac{-2(\alpha^n_0 + \sum_{i=1}^{k+1} \alpha^n_i \Delta_n - x_{k+1})(\alpha^n_0 + \sum_{i=1}^{k} \alpha^n_i \Delta_n - x_k)}{\Delta_n}\Big)\Big) \\ &\quad \times \frac{\exp\big(-\frac{(x_{k+1}-x_k)^2}{\Delta_n}\big)}{\sqrt{\pi \Delta_n}} dx_1 dx_2 \dots dx_m. \end{aligned} \tag{34}$$

Here, we use Equation (21) from Lemma 8 in the first equality as well as Equations (5) and (6) in the fifth equality. Finally, we can deduce Equation (32) by plugging Equation (34) and Equation (8) into Equation (33). □

With the same arguments as in the proofs of Lemmas 9, 10,11 and 12, we can define for any nonnegative integer $n \in \mathbb{N}$ the function H^n_0 as $H^n_0 : \mathbb{R}^+_* \to \mathbb{R}$ and the functions H^n_m as $H^n_m : \mathbb{R} \to \mathbb{R}$ for any nonnegative integer $m \in \{1, \dots, 2^n\}$. The definition of these functions is adapted from the definition of the functions G^n_m to the reflected Wiener process case. As the obtained equations are longer than in the Wiener process case, we do not report them.

The next lemma will be useful in showing the existence and unicity of the first coefficient α^n_0. More generally, this will be useful in the proof of Lemma 2. This basically states that the probability of the FPT started at time t^n_0 to a constant boundary function on the interval $[t^n_0, t^n_1]$, i.e. the function G^n_0 or the function H^n_0, is a strictly decreasing bijection from $\mathbb{R}^+_*$ to $(0, 1)$.

Lemma 13. *For any nonnegative integer $n \in \mathbb{N}$, we have that the function G^n_0 and the function H^n_0 are continuous and strictly decreasing bijections from the set $\mathbb{R}^+_*$ to the set $(0, 1)$.*

Proof. From Equation (22), we can see that the function G^n_0 is differentiable on $\mathbb{R}^+_*$ with negative derivatives for any nonnegative integer $n \in \mathbb{N}$. Thus, we have that the function G^n_0 is continuous and strictly decreasing. We also have that $G^n_0(\alpha) \to 1$ as $\alpha \to 0$ and $G^n_0(\alpha) \to 0$ as $\alpha \to \infty$, thus the function G^n_0 is a bijection from the set $\mathbb{R}^+_*$ to the set $(0, 1)$. Finally, we can prove the case H^n_0 with the same arguments. □

The next lemma is the counterpart of Lemma 13 when considering the function G^n_m and the function H^n_m for any nonnegative integer $n \in \mathbb{N}$ and any positive integer $m \in \{1, \dots, 2^n\}$.

Lemma 14. *For any nonnegative integer $n \in \mathbb{N}$ and any positive integer $m \in \{1, \dots, 2^n\}$, we have that the function G^n_m and the function H^n_m are continuous and strictly decreasing bijections from the set $\mathbb{R}$ to the set $(0, \int_{t^n_m}^{+\infty} f(s)ds)$.*

Proof. From Equations (25), (28) and (31), we can see that the function G^n_m is differentiable on the set $\mathbb{R}$ with negative derivative for any nonnegative integer $n \in \mathbb{N}$ and any positive integer $m \in \{1, \dots, 2^n\}$. Thus, we have that the function G^n_m is continuous and strictly decreasing. We also have that $G^n_m(\alpha) \to \int_{t^n_m}^{+\infty} f(s)ds$ as $\alpha \to -\infty$ and $G^n_m(\alpha) \to 0$ as $\alpha \to \infty$, thus the function G^n_m is a continuous bijection from the set $\mathbb{R}$ to the set $(0, \int_{t^n_m}^{+\infty} f(s)ds)$. Finally, we can prove the case H^n_m with the same arguments. □

The following lemma shows the positivity of the integral of the pdf f between two approximation times when we assume that Assumption 1 holds.

Lemma 15. *We assume that Assumption 1 holds. Then, we have for any nonnegative integer $n \in \mathbb{N}$ that*

$$0 < \int_0^{\delta_n \Delta_n} f(s)ds. \tag{35}$$

For any nonnegative integer $n \in \mathbb{N}$ and any positive integer $m \in \{1, \dots, 2^n\}$, we *also have that*

$$0 < \int_{t_{m-1}^n}^{t_m^n} f(s)ds. \tag{36}$$

Proof of Lemma 15. We define the supremum of the boundary absolute value g on the interval $[0, t_f]$ as

$$g_+ = \sup_{t \in [0, t_f]} \mid g(t) \mid .$$

By Assumption 1, we have that the boundary function g is continuous on the interval $[0, t_f]$. Since the interval $[0, t_f]$ is a compact space, it implies that $g_+ < \infty$. We have that $g \in \mathcal{G}$, thus $g_+ > 0$ by Definition 1. By Definition 2, we can deduce that $\mathrm{T}_g^Z \leq \mathrm{T}_{g_+}^Z$ a.s. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}_g^Z \in [0, \delta_n \Delta_n]). \tag{37}$$

Since $\mathbb{P}(\mathrm{T}_{g_+}^W \in [0, \delta_n \Delta_n]) = G_0^n(g_+)$ and $\mathbb{P}(\mathrm{T}_{g_+}^{|W|} \in [0, \delta_n \Delta_n]) = H_0^n(g_+)$, we obtain by Lemma 13 that $\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) > 0$. Then, we can deduce Equation (35) since f is equal to the density of T_g^Z by Equation (2) and Equation (4). Finally, Equation (36) follows by induction on the positive integer $m \in \{1, \dots, 2^n\}$ with similar arguments. □

We give now the proof of Lemma 2.

Proof of Lemma 2. For any nonnegative integer $n \in \mathbb{N}$, we prove Lemma 2 by induction on the nonnegative integer $m \in \{0, \dots, 2^n\}$. We start with the $m = 0$ case. Namely, we show that the first coefficient $\alpha_0^n \in \mathbb{R}_*^+$ is well-defined. We can deduce by Lemma 15 along with Assumption 1 that

$$0 < \int_0^{\delta_n \Delta_n} f(s)ds < 1. \tag{38}$$

From Expression (38) and Lemma 9, we can then deduce that

$$0 < G_0^n(\alpha_0^n) < 1 \text{ and } 0 < H_0^n(\alpha_0^n) < 1. \tag{39}$$

Finally, an application of the intermediate value theorem together with Lemma 13 and Expression (39) provides the existence and uniqueness of the first coefficient $\alpha_0^n \in \mathbb{R}_*^+$.

We consider now the $m > 0$ case. Namely, we show that the coefficient $\alpha_m^n \in \mathbb{R}$ is well-defined. By Lemma 15 along with Assumption 1, we get

$$0 < \int_{t_{m-1}^n}^{t_m^n} f(s)ds < \int_{t_{m-1}^n}^{+\infty} f(s)ds. \tag{40}$$

From Expression (40), Lemmas 10, 11 and 12, we can deduce that

$$0 < G_m^n(\alpha_m^n) < \int_{t_{m-1}^n}^{+\infty} f(s)ds \text{ and } 0 < H_m^n(\alpha_m^n) < \int_{t_{m-1}^n}^{+\infty} f(s)ds. \tag{41}$$

To conclude, an application of the intermediate value theorem along with Lemma 14 and Equation (41) provides the existence and uniqueness of the coefficient $\alpha_m^n \in \mathbb{R}$. □

6. Proofs of the main results

In this section, we give the proofs of the main results, namely Proposition 3 and Theorem 4. However, some ideas and notations used for the proofs of the main results were introduced in Section 5. Thus, the reader interested in the proofs of the main results should use Section 5 as a reference.

We first give the proof of Proposition 3. Namely, we show that the starting value of the piecewise-linear approximation converges to the starting value of the boundary function. The elementary idea of the proof consists in observing that the boundary function can be bounded below and above by positive constants for a very small time interval. Then, we show that these constants converge to the boundary starting value as the interval length goes to 0. This is possible since the starting value of the boundary function is positive and the boundary function is continuous with our assumptions.

Proof of Proposition 3. For any nonnegative integer $n \in \mathbb{N}$, we define the infimum of the boundary function g on the interval $[0, \delta_n \Delta_n]$ as

$$g_-^n(0) = \inf_{t \in [0, \delta_n \Delta_n]} g(t).$$

We also define the supremum of the boundary function g on the interval $[0, \delta_n \Delta_n]$ as

$$g_+^n(0) = \sup_{t \in [0, \delta_n \Delta_n]} g(t).$$

By Assumption 1, we have that the boundary function g is continuous on the interval $[0, \delta_n \Delta_n]$. Since the interval $[0, \delta_n \Delta_n]$ is a compact space, it implies that $-\infty < g_-^n(0) \leq g_+^n(0) < \infty$ for any nonnegative integer $n \in \mathbb{N}$. We have that $g \in \mathcal{G}$, thus $g_+^n(0) > 0$ for any nonnegative integer $n \in \mathbb{N}$ and $g_-^n(0) > 0$ for any positive integer n big enough by Definition 1. By Definition 2, we can deduce that $\mathrm{T}^Z_{g_-^n(0)} \leq \mathrm{T}^Z_g \leq \mathrm{T}^Z_{g_+^n(0)}$ a.s. for any positive integer n big enough.

Thus, we can deduce for any positive integer n big enough that

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_g \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n \Delta_n]).$$

By Equations (2) and (3), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f_g^Z(s) ds \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n \Delta_n]).$$

By Equation (4), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g_+^n(0)} \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f(s) ds \leq \mathbb{P}(\mathrm{T}^Z_{g_-^n(0)} \in [0, \delta_n \Delta_n]).$$

By Equation (7), the above inequalities can be reexpressed as

$$\mathbb{P}(\mathrm{T}^Z_{g^n_+(0)} \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_{\alpha^n_0} \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}^Z_{g^n_-(0)} \in [0, \delta_n \Delta_n]).$$

By Equation (22) and Lemma 9, the above inequalities can be reexpressed as

$$G^n_0(g^n_+(0)) \leq G^n_0(\alpha^n_0) \leq G^n_0(g^n_-(0)) \text{ or } H^n_0(g^n_+(0)) \leq H^n_0(\alpha^n_0) \leq H^n_0(g^n_-(0)). \quad (42)$$

Since we have that the function G^n_0 and the function H^n_0 are continuous and strictly decreasing bijections from the set $\mathbb{R}^+_*$ to the interval $(0, 1)$ by Lemma 13, we can invert the six sides of Expression (42) by G^n_0 when $Z = W$ and H^n_0 when $Z = |W|$. We obtain that $g^n_-(0) \leq \alpha^n_0 \leq g^n_+(0)$. By continuity of the boundary function g on the interval $[0, t_f]$ and Equation (5), we can conclude $\alpha^n_0 = g^n(0) \to g(0)$ as the number of intervals $n \to \infty$.

□

Now, we aim to prove Theorem 4 in what follows. Namely, we show that a subsequence of the piecewise-linear approximation uniformly converges to the boundary function when the length of each interval of linear approximation goes to 0 asymptotically. The proof goes in two steps. First, we show that the piecewise-linear approximation uniformly converges to some boundary function $\tilde{g} \in \mathcal{G}$ using Arzelà-Ascoli theorem on any compact space $[0, t_f]$. Second, we show that $\tilde{g}(t) = g(t)$ for any time $t \in [0, t_f]$.

We first give the definition of the piecewise linear boundary functions.

Definition 5. For any nonnegative integer $n \in \mathbb{N}$, we define the set of piecewise linear boundary functions as

$$\mathcal{G}^n = \Big\{ g \in \mathcal{G} \text{ s.t. } g \text{ is linear on each interval } \big[t^n_m, t^n_{m+1}\big] \text{ for } m \in \{0, \ldots, 2^n\} \Big\}.$$

In what follows, we give the definition of uniform boundedness.

Definition 6. The sequence of boundary functions $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly bounded if there is a constant $M > 0$ such that

$$\sup_{t \in [0, t_f], n \in \mathbb{N}} |g^n(t)| \leq M. \quad (43)$$

The following definition introduces the notion of uniform equicontinuity.

Definition 7. The sequence of boundary functions $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly equicontinuous if it satisfies the following property. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{t, s \in [0, t_f], |t - s| < \delta, n \in \mathbb{N}} |g^n(t) - g^n(s)| \leq \varepsilon. \quad (44)$$

We recall now the Arzelà-Ascoli theorem.

Theorem 16 (Arzelà-Ascoli theorem)**.** *We assume that the sequence of boundary functions $g^n \in \mathcal{G}^n$ defined on the interval $[0, t_f]$ is uniformly bounded in the sense of Definition 5 and uniformly equicontinuous in the sense of Definition 6. Then, there exists a subsequence which converges uniformly to some boundary function $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$.*

In the following proposition, we show that if we assume that the coefficients α_m^n are uniformly bounded, then the sequence of boundary functions g^n is uniformly bounded and uniformly equicontinuous.

Proposition 17. *We assume that Assumption 1 holds and that the coefficients α_m^n are uniformly bounded, namely*

$$\sup_{\substack{n\in\mathbb{N}\\ m=0,\cdots,2^n}} |\alpha_m^n| \leq K. \tag{45}$$

Then, the sequence of boundary functions g^n is uniformly bounded in the sense of Definition 5 and uniformly equicontinuous in the sense of Definition 6.

Proof. We start with the proof of Equation (43) from Definition 5. By algebraic manipulation, we can rewrite Equations (5) and (6) for any positive integer $m \in \{1, \dots, 2^n - 1\}$ as

$$g^n(u) \quad = \quad \alpha_0^n + \Delta_n \sum_{i=1}^{m} \alpha_i^n + \alpha_{m+1}^n (u - t_m^n) \text{ for any } u \in (t_m^n, t_{m+1}^n]. \tag{46}$$

We obtain that for any time $u \in (t_m^n, t_{m+1}^n]$ with $m \in \{1, \dots, 2^n - 1\}$ that

$$\begin{aligned}
|g^n(u)| &\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{m} |\alpha_i^n| + \left|\alpha_{m+1}^n\right| (u - t_m^n) \\
&\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{m+1} |\alpha_i^n| \\
&\leq |\alpha_0^n| + \Delta_n \sum_{i=1}^{2^n} |\alpha_i^n| \\
&\leq |\alpha_0^n| + t_f \sup_{\substack{n\in\mathbb{N}\\ i=1,\cdots,2^n}} |\alpha_i^n| \\
&\leq (1 + t_f) \sup_{\substack{n\in\mathbb{N}\\ i=0,\cdots,2^n}} |\alpha_i^n| \\
&\leq (1 + t_f)K.
\end{aligned}$$

Here, we use the triangular inequality in the first inequality and the fact that $u \in (t_m^n, t_{m+1}^n]$ in the second inequality. We also use the fact that $m \in \{1, \dots, 2^n - 1\}$ in the third equality and the definition of Δ_n in the fourth equality. Moreover, we use Equation (45) in the last inequality. We have thus shown that Equation (45) $\implies$ Equation (43).

We now prove Equation (44) from Definition 6. We consider any positive real number $\varepsilon > 0$. Accordingly, we set

$$\delta = \frac{\varepsilon}{2K}. \tag{47}$$

For any time $t \in [0, t_f]$, we define the corresponding index m_t^n such that $t \in [t^n_{m_t^n}, t^n_{m_t^n+1}]$. From Equation (46), we can deduce that

$$g^n(t) \quad = \quad \alpha_0^n + \Delta_n \sum_{i=1}^{m_t^n} \alpha_i^n + \alpha^n_{m_t^n+1}(t - t^n_{m_t^n}). \tag{48}$$

For any time $0 \leq s \leq t \leq t_f$ which satisfies

$$|t - s| < \delta, \tag{49}$$

we have that

$$\begin{aligned}
|g^n(t) - g^n(s)| &= \Big| \alpha_0^n + \Delta_n \sum_{i=1}^{m_t^n} \alpha_i^n + \alpha^n_{m_t^n+1}(t - t^n_{m_t^n}) - \\
&\quad (\alpha_0^n + \Delta_n \sum_{i=1}^{m_s^n} \alpha_i^n + \alpha^n_{m_s^n+1}(s - t^n_{m_s^n}))\Big| \\
&= \Big| \Delta_n \sum_{i=m_s^n}^{m_t^n} \alpha_i^n + \alpha^n_{m_t^n+1}(t - t^n_{m_t^n}) - \alpha^n_{m_s^n+1}(s - t^n_{m_s^n}) \Big| \\
&\leq |t - s| \sup_{\substack{n \in \mathbb{N} \\ i = 0, \cdots, 2^n}} |\alpha_i^n| \\
&\leq K \, |t - s| \, , \\
&\leq \varepsilon.
\end{aligned}$$

Here, we use Equation (48) in the first equality, algebraic manipulation in the second equality and the first inequality. We also use Equation (45) in the second inequality. In addition, we use Equation (47) and Expression (49) in the last inequality. We have thus shown that Equation (45) $\implies$ Equation (44). ☐

In the following proposition, we show that if we assume that Assumption 2 holds, then we have that the coefficients α_m^n are uniformly bounded.

Proposition 18. *We assume that Assumption 2 holds. Then, we have that the coefficients α_m^n are uniformly bounded, namely Equation (45) is satisfied.*

Proof. We define the supremum of the boundary derivative absolute value g' on the interval $[0, t_f]$ as

$$g'_+ = \sup_{t \in [0, t_f]} \mid g'(t) \mid .$$

Then, we define the bounding constant K as

$$K = 2 \sup(g_+, g'_+). \tag{50}$$

By Assumption 2, we have that the boundary function g is continuous on the interval $[0, t_f]$. Since the interval $[0, t_f]$ is a compact space, it implies that $g_+ <$

∞. We can also obtain by Assumption 2 that $g'_+ < \infty$. Thus, we can deduce that the bounding constant is finite, namely $K < \infty$.

Then, it is sufficient to show that Equation (45) is satisfied with K defined in Equation (50) to prove Proposition 18. For any nonnegative integer $n \in \mathbb{N}$, we consider a proof by induction on the nonnegative integer $m \in \{0, \dots, 2^n\}$. We start with the case $m = 0$, namely we show that $\alpha_0^n \leq K$. We have that $g \in \mathcal{G}$, thus $g_+ > 0$ by Definition 1. By Definition 2, we can deduce that $\mathrm{T}_g^Z \leq \mathrm{T}_{g_+}^Z$ a.s.

Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}_g^Z \in [0, \delta_n \Delta_n]).$$

By Equations (2) and (3), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f_g^Z(s) ds.$$

By Equation (4), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \int_0^{\delta_n \Delta_n} f(s) ds.$$

By Equation (7), the above inequality can be reexpressed as

$$\mathbb{P}(\mathrm{T}_{g_+}^Z \in [0, \delta_n \Delta_n]) \leq \mathbb{P}(\mathrm{T}_{\alpha_0^n}^Z \in [0, \delta_n \Delta_n]).$$

By Equation (22) and Lemma 9, the above inequality can be reexpressed as

$$G_0^n(g_+) \leq G_0^n(\alpha_0^n) \text{ or } H_0^n(g_+) \leq H_0^n(\alpha_0^n). \tag{51}$$

Since we have that G_0^n and H_0^n are continuous and strictly decreasing bijections from $\mathbb{R}_*^+$ to $(0, 1)$ by Lemma 13, we can invert the four sides of Expression (51) by G_0^n when $Z = W$ and H_0^n when $Z = |W|$. We can deduce that $\alpha_0^n \leq g_+$ which implies that $\alpha_0^n \leq K$.

We consider now the case $m = 1$, namely we show that $| \alpha_1^n | \leq K$. For any time $t \geq 0$, we define the linear boundary started at $g(0)$ with trend g'_+ and $-g'_+$ as respectively $\overline{g}(t) = g(0) + g'_+ t$ and $\underline{g}(t) = g(0) - g'_+ t$. By Definition 2, we can deduce that $\mathrm{T}_{\underline{g}}^Z \leq \mathrm{T}_g^Z \leq \mathrm{T}_{\overline{g}}^Z$ a.s. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}_{\overline{g}}^Z \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}_g^Z \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}_{\underline{g}}^Z \in [0, t_1^n]).$$

For any time $t \geq 0$, we define now the linear boundary started at α_0^n with trend K and $-K$ as respectively $\overline{g}^n(t) = \alpha_0^n + Kt$ and $\underline{g}^n(t) = \alpha_0^n - Kt$. When the positive integer n is big enough, we obtain that

$$\mathbb{P}(\mathrm{T}_{\overline{g}^n}^Z \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}_{g^n}^Z \in [0, t_1^n]) \leq \mathbb{P}(\mathrm{T}_{\underline{g}^n}^Z \in [0, t_1^n]).$$

By Equation (25) and Lemma 10, the above inequalities can be reexpressed as

$$G_1^n(K) \leq G_1^n(\alpha_1^n) \leq G_1^n(-K) \text{ or } H_1^n(K) \leq H_1^n(\alpha_1^n) \leq H_1^n(-K). \tag{52}$$

By Lemma 14, we have that the function G_1^n and the function H_1^n are continuous and strictly decreasing bijections from the set $\mathbb{R}$ to the interval $(0, \int_{t_1^n}^{+\infty} f(s)ds)$. Thus, we can invert the six sides of Expression (52) by the function G_1^n when $Z = W$ and the function H_1^n when $Z = |W|$. Finally, we can deduce that $| \alpha_1^n | \leq g'_+$ which implies $| \alpha_1^n | \leq K$.

We consider now the case $m = 2$, namely we show that $| \alpha_2^n | \leq K$. We introduce the boundary function $\overline{g}(t)$ which is equal to the boundary function g on the interval $[0, t_1^n]$ and linear with trend g'_+ for any time $t \geq t_1^n$. More specifically, it is defined as $\overline{g}(t) = g(t)$ for any time $t \in [0, t_1^n]$ and $\overline{g}(t) = g(t_1^n) + g'_+(t - t_1^n)$ for any time $t \geq t_1^n$. We also introduce the boundary function $\underline{g}(t)$ which is equal to the boundary function g on the interval $[0, t_1^n]$ and linear with trend $-g'_+$ for any time $t \geq t_1^n$. More specifically, it is defined as $\underline{g}(t) = g(t)$ for any time $t \in [0, t_1^n]$ and $\underline{g}(t) = g(t_1^n) - g'_+(t - t_1^n)$ for any time $t \geq t_1^n$.

By Definition 2, we can deduce that $\mathrm{T}^Z_{\underline{g}} \leq \mathrm{T}^Z_g \leq \mathrm{T}^Z_{\overline{g}}$ a.s. Thus, we can deduce that

$$\mathbb{P}(\mathrm{T}^Z_{\overline{g}} \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}^Z_g \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}^Z_{\underline{g}} \in [t_1^n, t_2^n]).$$

Then, we introduce the boundary function $\overline{g}^n(t)$ which is equal to the boundary function g^n on $[0, t_1^n]$ and linear with trend K for any time $t \geq t_1^n$. It is defined as $\overline{g}^n(t) = g^n(t)$ for any time $t \in [0, t_1^n]$ and $\overline{g}^n(t) = g(t_1^n) + K(t - t_1^n)$ for any time $t \geq t_1^n$. We also introduce the boundary function $\underline{g}^n(t)$ which is equal to the boundary function g^n on the interval $[0, t_1^n]$ and linear with trend $-K$ for any time $t \geq t_1^n$. This is defined as $\underline{g}^n(t) = g^n(t)$ for any time $t \in [0, t_1^n]$ and $\underline{g}^n(t) = g^n(t_1^n) - K(t - t_1^n)$ for any time $t \geq t_1^n$.

When the positive integer n is big enough, we obtain that

$$\mathbb{P}(\mathrm{T}^Z_{\overline{g}^n} \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}^Z_{g^n} \in [t_1^n, t_2^n]) \leq \mathbb{P}(\mathrm{T}^Z_{\underline{g}^n} \in [t_1^n, t_2^n]).$$

By Equation (28) and Lemma 11, the above inequalities can be reexpressed as

$$G_2^n(K) \leq G_2^n(\alpha_2^n) \leq G_2^n(-K) \text{ or } H_2^n(K) \leq H_2^n(\alpha_2^n) \leq H_2^n(-K). \tag{53}$$

We have by Lemma 14 that the function G_2^n and the function H_2^n are continuous and strictly decreasing bijections from the set $\mathbb{R}$ to the interval $(0, \int_{t_2^n}^{+\infty} f(s)ds)$. Thus, we can invert the six sides of Expression (53) by the function G_2^n when $Z = W$ and the function H_2^n when $Z = |W|$. Moreover, we can deduce that $| \alpha_2^n | \leq K$. Finally, the case with the positive integer $m > 2$ follows with similar arguments. □

The following corollary is an application of Arzelà-Ascoli theorem.

Corollary 19. *We assume that Assumption 2 holds. Then, there exists a subsequence g^{n_k} of the piecewise-linear approximation g^n which converges uniformly to some boundary function $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$.*

Proof. This is an application of Theorem 16 along with Propositions 17 and 18. □

In the lemma that follows, we give a.s. convergence of the random variable $T^Z_{h^n}\mathbf{1}_{\{T^Z_{h^n}\leq t_f\}}$ to the random variable $T^Z_h\mathbf{1}_{\{T^Z_h\leq t_f\}}$ when the piecewise-linear approximation h^n converges uniformly to the boundary function h on the interval $[0,t_f]$. For the proof of Theorem 4, we only need the convergence in distribution.

Lemma 20. *For any sequence of piecewise-linear approximation $h^n \in \mathcal{G}^n$ which converges uniformly on the interval $[0,t_f]$ to some boundary function $h\in\mathcal{G}$ satisfying Assumption 1, we have that the random variable $\mathrm{T}^Z_{h^n}\mathbf{1}_{\{\mathrm{T}^Z_{h^n}\leq t_f\}}$ converges a.s. to the random variable $\mathrm{T}^Z_h\mathbf{1}_{\{\mathrm{T}^Z_h\leq t_f\}}$. As a consequence, we deduce that the random variable $\mathrm{T}^Z_{h^n}\mathbf{1}_{\{\mathrm{T}^Z_{h^n}\leq t_f\}}$ converges in distribution to the random variable $\mathrm{T}^Z_h\mathbf{1}_{\{\mathrm{T}^Z_h\leq t_f\}}$.*

Proof. To prove that the random variable $\mathrm{T}^Z_{h^n}\mathbf{1}_{\{\mathrm{T}^Z_{h^n}\leq t_f\}}$ converges a.s. to the random variable $\mathrm{T}^Z_h\mathbf{1}_{\{\mathrm{T}^Z_h\leq t_f\}}$, it is sufficient to show that for any positive real number $\epsilon>0$ there exists a positive integer $N_\epsilon\in\mathbb{N}$ such that for any positive integer $n\in\mathbb{N}_*$ satisfying $n\geq N_\epsilon$ we have a.s.

$$\left|\mathrm{T}^Z_{h^n}\mathbf{1}_{\{\mathrm{T}^Z_{h^n}\leq t_f\}} - \mathrm{T}^Z_h\mathbf{1}_{\{\mathrm{T}^Z_h\leq t_f\}}\right| \leq \epsilon. \tag{54}$$

As the piecewise-linear approximation h^n converges uniformly to the boundary function h on the interval $[0,t_f]$, we can deduce the following property. For any positive real number $\epsilon_h>0$, there exists a nonnegative integer $N_{\epsilon_h}\in\mathbb{N}$ such that for any positive integer $n\in\mathbb{N}_*$ with $n\geq N_{\epsilon_h}$ we have

$$\sup_{t\in[0,t_f]} \mid h^n(t)-h(t)\mid\leq \epsilon_h. \tag{55}$$

Then, we set the value of the random variable ϵ_h as

$$\epsilon_h = \frac{1}{2}\sup_{\mathrm{T}^Z_h\leq t\leq \mathrm{T}^Z_h+\epsilon\leq t_f}\left|Z_t-h(t)\right|. \tag{56}$$

First, we can see that the random variable ϵ_h defined in Equation (56) is positive. Second, we have that a.s. the stochastic process Z_t first hits the piecewise-linear approximation h^n on the interval $[\mathrm{T}^Z_h-\epsilon, T^Z_h+\epsilon]$, namely we have shown that $\mathrm{T}^Z_{h^n}\in[\mathrm{T}^Z_h-\epsilon,\mathrm{T}^Z_h+\epsilon]$ whenever Expression (55) holds with ϵ_h from Equation (56). Thus, we have shown Expression (54) with $N_\epsilon=N_{\epsilon_h}$. □

We now consider a discretization length in the order Δ_n so that we obtain that the time discretization is nested. Namely, for any time t^n_m and any positive integer $l\geq m$ there exists a time t^l_k such that $t^n_m=t^l_k$. This is required to prove the following lemma which in turn will be used to prove that the limit of a subsequence obtained by Arzelà-Ascoli theorem satisfies Equation (4).

Lemma 21. *We assume that Assumption 1 holds. For any nonnegative integer $n\in\mathbb{N}$, any nonnegative integer $l\in\mathbb{N}$ satisfying $l\geq n$ and any nonnegative integer $m\in\{0,\ldots,2^n\}$, the piecewise-linear approximation satisfies*

$$\mathbb{P}\left(\mathrm{T}^Z_{g^l}\in[t^n_m,t^n_{m+1}]\right)=\int_{t^n_m}^{t^n_{m+1}} f(s)ds. \tag{57}$$

Proof. For any nonnegative integer $n \in \mathbb{N}$, any nonnegative integer $l \in \mathbb{N}$ satisfying $l \geq n$ and any nonnegative integer $m \in \{0, \dots, 2^n\}$, we have

$$\begin{aligned}\mathbb{P}\left(\mathrm{T}^Z_{g^l} \in [t^n_m, t^n_{m+1}]\right) &= \sum_{\substack{i\in\mathbb{N} \text{ s.t.} \\ t^n_m \leq t^l_i \leq t^l_{i+1} \leq t^n_{m+1}}} \mathbb{P}\left(\mathrm{T}^Z_{g^l} \in [t^l_i, t^l_{i+1}]\right) \\ &= \sum_{\substack{i\in\mathbb{N} \text{ s.t.} \\ t^n_m \leq t^l_i \leq t^l_{i+1} \leq t^n_{m+1}}} \int_{t^l_i}^{t^l_{i+1}} f(s)ds \\ &= \int_{t^n_m}^{t^n_{m+1}} f(s)ds.\end{aligned}$$

Here, we use the fact that

$$[t^n_m, t^n_{m+1}] = \bigcup_{\substack{i\in\mathbb{N} \text{ s.t.} \\ t^n_m \leq t^l_i \leq t^l_{i+1} \leq t^n_{m+1}}} [t^l_i, t^l_{i+1}]$$

since the time discretization is nested in the first equality. We also use Equations (7) and (8) in the second equality. □

We provide in what follows the proof of Theorem 4 which is the main result of our paper. This shows that a subsequence of the new approximation uniformly converges to the boundary when the length of each interval of linear approximation goes to 0 asymptotically. This proof is based on an application of previously obtained results and shows that the boundary function $\tilde{g}(t) = g(t)$ for any $t \in [0, t_f]$.

Proof of Theorem 4. By Corollary 19 along with Assumption 2, there exists a subsequence g^{n_k} of the piecewise-linear approximation g^n which converges uniformly to some boundary function $\tilde{g} \in \mathcal{G}$ defined on the interval $[0, t_f]$. We first show that $f^Z_{\tilde{g}}(t) = f(t)$ for any time $t \in [0, t_f]$. By Borel arguments, it is sufficient to show that for any nonnegative integer $p \in \mathbb{N}$ and any nonnegative integer $k \in \{0, \dots, 2^p - 1\}$ we have

$$\mathbb{P}\big(\mathrm{T}^Z_{\tilde{g}} \in [k\Delta_p, (k+1)\Delta_p]\big) = \int_{k\Delta_p}^{(k+1)\Delta_p} f(s)ds. \tag{58}$$

Then, we have that

$$\begin{aligned}\mathbb{P}\big(\mathrm{T}^Z_{\tilde{g}} \in [k\Delta_p, (k+1)\Delta_p]\big) &= \lim_{n\to\infty} \mathbb{P}\big(\mathrm{T}^Z_{g^{n_k}} \in [k\Delta_p, (k+1)\Delta_p]\big) \\ &= \int_{k\Delta_p}^{(k+1)\Delta_p} f(s)ds.\end{aligned}$$

Here, the first equality corresponds to the convergence in distribution of the random variable $\mathrm{T}^Z_{h^n}\mathbf{1}_{\{\mathrm{T}^Z_{h^n} \leq t_f\}}$ to the random variable $\mathrm{T}^Z_h\mathbf{1}_{\{\mathrm{T}^Z_h \leq t_f\}}$ by Lemma

20 along with Assumption 2. Moreover, we use Lemma 21 along with Assumption 2 in the second equality. Thus, we have shown Equation (58), which implies that $f_{\tilde{g}}^{Z}(t) = f(t)$ for any time $t \in [0, t_f]$. Since there is uniqueness of the IFPT problem by the related papers mentioned in the introduction, we can deduce that $\tilde{g}(t) = g(t)$ for any time $t \in [0, t_f]$. □

7. Conclusion

In this paper, we have studied the inverse first-passage time problem. The problem determines a boundary function such that the first-passage time of a Wiener process to this boundary function has a given distribution. An approximation of the boundary function by a piecewise-linear boundary was given by equating the probability of the first-passage time to a linear boundary and the increment of the distribution on each interval. This was based on the starting value of the boundary function, which is unknown in practice. We have proposed an approximation for the starting value of the boundary function. We have considered asymptotics where the length of each interval goes to 0.

We have first showed that the approximation for the starting value of the boundary function converges to the starting value of the boundary function when assuming that the boundary function is absolutely continuous and with positive starting value. We have also showed that a subsequence of the piecewise-linear approximation uniformly converges to the boundary function. The proofs were based on an application of Arzelà-Ascoli theorem. A numerical study have shown that the piecewise-linear approximation is sensitive to the starting value of the boundary function and the starting value of the boundary function derivative. The results obtained in the numerical study indicated that the piecewise-linear approximation is adequate and relatively safe to use in practice.

One limitation in this paper is that we have only obtained the convergence for a subsequence of the piecewise-linear approximation. The reason is that we were not able to prove directly the convergence by extending the proving techniques used in Zucca and Sacerdote (2009). More specifically, their two main ideas are the use of concavity inequalities and the implicit function theorem. Assuming that the FPT cdf is absolutely continuous and the boundary is monotone concave, they prove in Theorem 4.3 (p. 1331) that the error due to the approximation is of the order equal to the maximum of the initial error and the squared interval length. We were able to weaken their assumptions on concavity by assumptions on differentiability with uniformly bounded derivatives. The elementary idea of the proof consisted in bounding the difference between the approximation and the boundary value by a linear function on each interval. However, we were not able to extend their direct use of the implicit function theorem with the new asymptotics. The reason is that we need to use the implicit function theorem with an increasing number of intervals, whereas they only use it with a finite number of intervals. Although we were not able to track down the calculation, we conjecture that the direct convergence also holds. This extension is left for future work.

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References

Abundo, M. (2006). Limit at zero of the first-passage time density and the inverse problem for one-dimensional diffusions. *Stochastic analysis and applications* **24** 1119–1145.

Anderson, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Annals of Mathematical Statistics* **31** 165–197.

Anulova, S. (1981). On Markov stopping times with a given distribution for a Wiener process. *Theory of Probability & Its Applications* **25** 362–366.

Avellaneda, M. and Zhu, J. (2001). Distance to default. *Risk* **14** 125–129.

Beiglböck, M., Eder, M., Elgert, C. and Schmock, U. (2018). Geometry of distribution-constrained optimal stopping problems. *Probability Theory and Related Fields* **172** 71–101.

Breiman, L. (1967). First exit times from a square root boundary. In *Proceedings of the 5th Berkeley Symposyum Mathematical Statistics* **5** 9–17. University of California Press.

Buonocore, A., Nobile, A. G. and Ricciardi, L. M. (1987). A new integral equation for the evaluation of first-passage-time probability densities. *Advances in Applied Probability* **19** 784–800.

Chen, X., Chadam, J. and Saunders, D. (2022). Higher-order regularity of the free boundary in the inverse first-passage problem. *SIAM Journal on Mathematical Analysis* **54** 4695–4720.

Chen, X., Cheng, L., Chadam, J. and Saunders, D. (2011). Existence and uniqueness of solutions to the inverse boundary crossing problem for diffusions. *Annals of Applied Probability* **21** 1663–1693.

Cheng, L., Chen, X., Chadam, J. and Saunders, D. (2006). Analysis of an inverse first passage problem from risk management. *SIAM Journal on Mathematical Analysis* **38** 845–873.

Daniels, H. E. (1969). The minimum of a stationary Markov process superimposed on a U-shaped trend. *Journal of Applied Probability* **6** 399–408.

Doob, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Annals of Mathematical Statistics* **20** 393–403.

Dudley, R. M. and Gutmann, S. (1977). Stopping times with given laws. *Séminaire de Probabilités de Strasbourg* **11** 51–58.

Durbin, J. (1971). Boundary-crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of the Kolmogorov-Smirnov test. *Journal of Applied Probability* **8** 431–453.

Ekström, E. and Janson, S. (2016). The inverse first-passage problem and optimal stopping. *Annals of Applied Probability* **26** 3154–3177.

Fukasawa, M. and Obloj, J. (2020). Efficient discretisation of stochastic differential equations. *Stochastics* **92** 833–851.

Hull, J. C. and White, A. D. (2001). Valuing credit default swaps II: Modeling default correlations. *The Journal of derivatives* **8** 12–21.

Iscoe, I. and Kreinin, A. (1999). Default boundary problem Technical Report, Technical report, Algorithmics Inc.

Iscoe, I., Kreinin, A. and Rosen, D. (1999). An integrated market and credit risk portfolio model. *Algo Research Quarterly* **2** 21–38.

Jacod, J. and Shiryaev, A. (2003). *Limit theorems for stochastic processes*, 2nd ed. Berlin: Springer-Verlag.

Jaimungal, S., Kreinin, A. and Valov, A. (2014). The generalized Shiryaev problem and Skorokhod embedding. *Theory of Probability & Its Applications* **58** 493–502.

Klump, A. and Kolb, M. (2023). Uniqueness of the Inverse First-Passage Time Problem and the Shape of the Shiryaev Boundary. *Theory of Probability & Its Applications* **67** 570–592.

Klump, A. and Kolb, M. (2024). An elementary approach to the inverse first-passage-time problem for soft-killed Brownian motion. *Journal of Applied Probability* **61** 279–300.

Klump, A. and Savov, M. (2025). Conditions for existence and uniqueness of the inverse first-passage time problem applicable for Lévy processes and diffusions. *Annals of Applied Probability* **35** 1791–1827.

Malmquist, S. (1954). On certain confidence contours for distribution functions. *Annals of Mathematical Statistics* **25** 523–533.

Peskir, G. (2002a). On integral equations arising in the first-passage problem for Brownian motion. *The Journal of Integral Equations and Applications* **14** 397–423.

Peskir, G. (2002b). Limit at zero of the Brownian first-passage density. *Probability Theory and Related Fields* **124** 100–111.

Potiron, Y. (2025a). Non-explicit formula of boundary crossing probabilities by the Girsanov theorem. *Annals of the Institue of Statistical Mathematics* **77** 353–385.

Potiron, Y. (2025b). First passage time and inverse problem for continuous local martingales. *Revise and Resubmit for the Journal of Statistical Planning and Inference.*

Schmidt, T. and Novikov, A. (2008). A structural model with unobserved default boundary. *Applied Mathematical Finance* **15** 183–203.

Song, J.-S. and Zipkin, P. (2011). An approximation for the inverse first passage time problem. *Advances in Applied Probability* **43** 264–275.

Song, J.-S. and Zipkin, P. (2013). Supply streams. *Manufacturing & Service Operations Management* **15** 444–457.

Strassen, V. (1967). Almost sure behavior of sums of independent random variables and martingales. In *Proceedings of the 5th Berkeley Symposyum Mathematical Statistics and Probability* **2** 315–343.

Wang, L. and Pötzelberger, K. (1997). Boundary crossing probability for Brownian motion and general boundaries. *Journal of Applied Probability* **34**

54–65.
Zucca, C. and Sacerdote, L. (2009). On the inverse first-passage-time problem for a Wiener process. *Annals of Applied Probability* **19** 1319–1346.