Supplement of Non-explicit formula of boundary crossing probabilities by the Girsanov theorem

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This is the "supplementary material" of "Non-explicit formula of boundary crossing probabilities by the Girsanov theorem" by Yoann Potiron published in the Annals of the Institute of Statistical Mathematics. Supplement A gives the results in the two-sided stochastic boundary process case. Supplement B collects the proofs in the two-sided stochastic boundary process case.

Appendix A: Results in the two-sided stochastic boundary process case

In this appendix, we consider the case when the two-sided boundary and the drift are stochastic processes and the variance is random.

We first give the definition of the set of stochastic boundary processes.

Definition 1. We define the set of stochastic two-sided boundary processes as $\mathcal{J} = \mathbb{R}^+ \times \Omega \to \mathbb{R}^2$ such that for any $(g,h) \in \mathcal{J}$ and $\omega \in \Omega$ we have $(g,h)(\omega) \in \mathcal{I}$ as well as g and h are $\mathbf{F}$-adapted.

We now give the definition of the FPT.

Definition 2. We define the FPT of an $\mathbf{F}$-adapted continuous process Z to the two-sided boundary $(g,h) \in \mathcal{J}$ satisfying $g_0 \leq Z_0 \leq h_0 \ \forall \omega \in \Omega$ as

$$\mathrm{T}_{g,h}^{Z} = \inf\{t \in \mathbb{R}^+ \text{ s.t. } Z_t \geq g_t \text{ or } Z_t \leq h_t\}. \quad (1)$$

We can rewrite $\mathrm{T}_{g,h}^{Z}$ as the infimum of two $\mathbf{F}$-stopping times, i.e., $\mathrm{T}_{g,h}^{Z} = \inf(\mathrm{T}_{h}^{Z}, \mathrm{T}_{-g}^{-Z})$. Thus, it is an $\mathbf{F}$-stopping time. We can rewrite the boundary crossing probability $P_{g,h}^{Z}$ as the cdf of $\mathrm{T}_{g,h}^{Z}$, i.e.,

$$P_{g,h}^{Z}(t) = \mathbb{P}(\mathrm{T}_{g,h}^{Z} \leq t) \text{ for any } t \geq 0. \quad (2)$$

We assume that μ is an $\mathbf{F}$-adapted stochastic process which satisfies $\mathbb{P}(g_0 < \mu_0 < h_0) = 1$. We also assume that the variance σ^2 is time-invariant, random, and such that $\mathbb{P}(\sigma^2 = 0) = 0$. Finally, we assume that v is independent of W where v is defined as $v = (g, h, \mu, \sigma)$.

Assumption D. We assume that $\mathbb{P}(\exists t \in [0,T] \text{ s.t } u_t \neq 0) = 1$. We also assume that u is absolutely continuous on $[0,T]$, i.e., there exists a stochastic process $\theta : [0,T] \times \Omega \to \mathbb{R}$ with $u_t = \int_0^t \theta_s ds$, a.s.. Finally, we assume that $\mathbb{E}[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)] < \infty$.

By **Assumption D**, M satisfies Novikov's condition and thus is a positive martingale.

Lemma 1. *Under **Assumption D**, we have that M is a positive martingale. Thus, we can consider an equivalent probability measure $\mathbb{Q}$ such that the Radon-Nikodym derivative is defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$. Finally, Y is a standard Wiener process under $\mathbb{Q}$.*

The elementary idea in this appendix is to condition by both W_T and v, i.e., to derive results of the form $\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v)$. The next proposition reexpresses $\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v)$ under $\mathbb{Q}$. We define $\overline{W}_t$ as

$$\overline{W}_t = \int_0^t \theta_s dW_s. \tag{3}$$

Proposition 1. *Under **Assumption D**, we have*

$$\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v) \quad = \quad \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} M_T^{-1} | W_T, v\big]. \tag{4}$$

This can be reexpressed as

$$\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T | W_T, v) = \tag{5}$$
$$\mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} | W_T, \overline{W}_T, v\big] | W_T, v\big].$$

We define the correlation under $\mathbb{P}$ between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ as ρ, i.e., $\rho = \mathrm{Cor}_{\mathbb{P}}(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}})$.

Lemma 2. *Under **Assumption D**, we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$. We can also show that $\rho = \frac{1}{T}\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s ds}{\sqrt{\int_0^T \theta_s^2 ds}}\Big]$ a.s.. Moreover, there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$, which is* *independent of* W_T*, and such that* $\overline{W}_T$ *when normalized can be reexpressed a.s. as*

$$\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}} = \rho \frac{W_T}{\sqrt{T}} + \sqrt{1-\rho^2}\widetilde{W}. \tag{6}$$

This can be reexpressed a.s. as

$$\overline{W}_T = \alpha W_T + \widetilde{\alpha}\widetilde{W}, \tag{7}$$

where $\alpha = \rho\sqrt{T^{-1}\int_0^T \theta_s^2 ds}$ *a.s. and* $\widetilde{\alpha} = \sqrt{(1-\rho^2)\int_0^T \theta_s^2 ds}$ *a.s.. If we define* $\widetilde{\theta}_t = \frac{\theta_s - \alpha}{\widetilde{\alpha}}$*, we can reexpress* $\widetilde{W}$ *a.s. as*

$$\widetilde{W} \quad = \quad \int_0^T \widetilde{\theta}_s dW_s. \tag{8}$$

Moreover, $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ *is a standard normal variable under* $\mathbb{Q}$*. Finally, the conditional distribution of* $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ *given* v*, i.e.,* $\mathcal{D}(\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds|v)$*, is standard normal under* $\mathbb{Q}$*.*

Our main result is the next theorem.

Theorem 1. *Under **Assumption D**, we have*

$$\begin{aligned}\mathbb{P}(\mathrm{T}^Y_{b,c} \leq T|W_T, v) \quad &= \quad \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ &\quad \times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c} \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big].\end{aligned} \tag{9}$$

We first calculate $\mathbb{Q}(\mathrm{T}^Y_{b,c} \leq T|W_T, v)$.

Lemma 3. *Under **Assumption D**, we have*

$$\mathbb{Q}(\mathrm{T}^Y_{b,c} \leq T|W_T, v) \quad = \quad \sum_{j=1}^{\infty} q^Y_{b,c}(j|Y_T)\mathbf{1}_{\{Y_T \in [c_T, b_T]\}} + \mathbf{1}_{\{Y_T \notin [c_T, b_T]\}}. \tag{10}$$

The next theorem gives a formula based on the strong theoretical assumption (11).

Theorem 2. *We assume that **Assumption D** and the following assumption*

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c} \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big] \\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}^Y_{b,c} \leq T\}}|W_T, v\big]\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big]\end{aligned} \tag{11}$$

holds. Then, we have

$$\mathbb{P}(\mathrm{T}^Y_{b,c} \leq T|W_T, v) \quad = \quad \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \tag{12}$$

$$\times\Big(\sum_{j=1}^{\infty} q_{b,c}^{Y}(j|Y_T)\mathbf{1}_{\{Y_T\in[c_T,b_T]\}} +\mathbf{1}_{\{Y_T\notin[c_T,b_T]\}}\Big)\exp\big(\widetilde{\alpha}\int_0^T\widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \quad (13)$$

Finally, we get $P_b^Y(T)$ in the next corollary, by integrating $\mathbb{P}(\mathrm{T}_b^Y \le T|W_T,v)$ with respect to the value of (W_T,v). We define the arrival space and cdf of v as respectively Π_v and P_v. Moreover, we define y_u, y_b, y_θ, etc. following the above definitions when integrating with respect to $y\in\Pi_v$.

Corollary 1. *Under **Assumption D**, we have*

$$\begin{aligned} P_{b,c}^Y(T) &= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\ &+\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big) \\ &\times\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\le T\}}\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T=x,v=y\big]dxdP_v(y). \end{aligned} \quad (14)$$

If we further assume (11), we have

$$\begin{aligned} P_{b,c}^Y(T) &= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\ &+\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v}\frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)\exp\big(-y_\alpha x+\frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big) \\ &\times\Big(\sum_{j=1}^{\infty} y_{q,y_b,y_c}^{x+y_{u,T}}(j|x+y_{u,T})\mathbf{1}_{\{x\in[y_{c,T}-y_{u,T},y_{b,T}-y_{u,T}]\}} \\ &+\mathbf{1}_{\{x\notin[y_{c,T}-y_{u,T},y_{b,T}-y_{u,T}]\}}\Big) \\ &\times\exp\big(y_{\widetilde{\alpha}}\int_0^T y_{\widetilde{\theta},s}y_{\theta,s}ds\big)\mathcal{L}_N(y_{\widetilde{\alpha}})dxdP_v(y). \end{aligned} \quad (15)$$

Appendix B: Proofs in the two-sided stochastic boundary process case

In this section, we consider the proofs in the case when the two-sided boundary and the drift are stochastic processes and the variance is random.

The elementary idea in the proofs of this section is to condition by both W_T and v, i.e., to derive results of the form $\mathbb{P}(\mathrm{T}_{b,c}^Y \le T|W_T,v)$. The proof of Proposition 1 is based on Lemma 1.

Proof of Proposition 1. By definition of the conditional probability, Equation (4) can be rewritten formally as

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y\leq T\}}|W_T,v\big] \quad = \quad \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}M_T^{-1}|W_T,v\big]. \tag{16}$$

For any $\sigma(W_T,v)$-measurable event E_T, we can use a change of probability in the expectation by Lemma 1, along with **Assumption D**, and we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}\big] \quad = \quad \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}M_T^{-1}\mathbf{1}_{E_T}\big]. \tag{17}$$

We can deduce Equation (16) from Equation (17) by definition of the conditional expectation. By definition of the conditional probability, Equation (5) can be rewritten formally as

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}|W_T,v\big]\\ &= \mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}|W_T,\overline{W}_T,v\big]|W_T,v\big].\end{aligned} \tag{18}$$

By definition of the conditional expectation, we can deduce what follows. If we can show that for any E_T, which is $\sigma(W_T,v)$-measurable, that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}\big]\\ &= \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{Q}}\big[M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}|W_T,\overline{W}_T,v\big]|W_T,v\big]\mathbf{1}_{E_T}\Big],\end{aligned} \tag{19}$$

then Equation (18) holds. Let E_T be a $\sigma(W_T,v)$-measurable event. By Lemma 1 along with **Assumption D**, we obtain that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}\big] \quad = \quad \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}M_T^{-1}\big]. \tag{20}$$

Then, we have by the law of total expectation that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}M_T^{-1}\big] =\\ &\mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}M_T^{-1}|W_T,\overline{W}_T,v\big]\big].\end{aligned} \tag{21}$$

Since $\mathbf{1}_{E_T}$ and M_T^{-1} are $\sigma(W_T,\overline{W}_T,v)$-measurable random variables, we can pull them out of the conditional expectation and deduce that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}\mathbf{1}_{E_T}M_T^{-1}|W_T,\overline{W}_T,v\big]\big]\\ &= \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{E_T}M_T^{-1}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y\leq T\}}|W_T,\overline{W}_T,v\big]\big].\end{aligned} \tag{22}$$

If we use Equations (20)-(21)-(22), we can deduce that Equation (19) holds. □

In what follows, we give the proof of Lemma 2.

Proof of Lemma 2. By **Assumption D**, we can deduce that $0 < \int_0^T \theta_s^2 ds < \infty$ a.s.. Thus, we can normalize $\overline{W}_T$ by $\sqrt{\int_0^T \theta_s^2 ds}$ a.s. and we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a mixed normal random variable a.s. by definition. Using the same arguments from the proof of Lemma 2.10, we have that its conditional mean under $\mathbb{P}$ is a.s. equal to

$$\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big] = 0.$$

We also have that its conditional variance under $\mathbb{P}$ is a.s. equal to

$$\operatorname{Var}_{\mathbb{P}}\Big(\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big) = 1. \tag{23}$$

Since its conditional mean and conditional variance are nonrandom, we obtain that its mean under $\mathbb{P}$ is equal to $\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s dW_s}{\sqrt{\int_0^T \theta_s^2 ds}}\Big| v\Big]\Big] = 0$ by the the law of total expectation and Equation (23). Similarly, we obtain that its variance is equal to 1 by the law of total expectation and Equation (23). Thus, we have that $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ is a standard normal random variable under $\mathbb{P}$. Since $(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}})$ is a centered normal random vector under $\mathbb{P}$, there exists a standard normal random variable $\widetilde{W}$ under $\mathbb{P}$ which is independent of W_T and such that Equation (6) holds. Then, we can calculate that the covariance between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ under $\mathbb{P}$ is equal to

$$\operatorname{Cov}_{\mathbb{P}}\Big(W_T, \frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}\Big) = \mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s ds}{\sqrt{\int_0^T \theta_s^2 ds}}\Big]. \tag{24}$$

Now, we can calculate that the correlation between W_T and $\frac{\overline{W}_T}{\sqrt{\int_0^T \theta_s^2 ds}}$ under $\mathbb{P}$ is equal to

$$\rho = \frac{1}{T}\mathbb{E}_{\mathbb{P}}\Big[\frac{\int_0^T \theta_s ds}{\sqrt{\int_0^T \theta_s^2 ds}}\Big].$$

Equation (7) can be deduced directly from Equation (6). Moreover, we can reexpress $\widetilde{W}$ as

$$\widetilde{W} = \int_0^T \widetilde{\theta}_s dW_s.$$

Moreover, we can deduce that $\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds$ is a standard normal variable under $\mathbb{Q}$. This is due to its expression (8) and since by Lemma 1 along with **Assumption D**, Y is a Wiener process under $\mathbb{Q}$. Finally, $\mathcal{D}(\widetilde{W} + \int_0^T \widetilde{\theta}_s \theta_s ds|v)$ is standard normal under $\mathbb{Q}$ by Equation (8). □

We provide now the proof of Theorem 1, which is based on Lemma 2.

Proof of Theorem 1. Using the same arguments from the proof of Theorem 2.11, we can reexpress M_T as

$$M_T \quad = \quad \exp\big(\alpha W_T - \frac{1}{2}\int_0^T \theta_s^2 ds\big)\exp\big(\widetilde{\alpha}\widetilde{W}\big).$$

Then, we have

$$\begin{aligned}\mathbb{P}(\mathrm{T}_{b,c}^Y \leq T|W_T, v) \quad &= \quad \exp\big(-\alpha W_T + \frac{1}{2}\int_0^T \theta_s^2 ds\big) \\ &\times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big].\end{aligned}$$

Thus, we have shown Equation (9). □

We now give the proof of Lemma 3.

Proof of Lemma 3. By definition of the conditional probability, Equation (10) can be rewritten formally as

$$\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}}|W_T\big] \quad = \quad \sum_{j=1}^{\infty} q_{b,c}^Y(j|Y_T)\mathbf{1}_{\{Y_T \in [c_T, b_T]\}} + \mathbf{1}_{\{Y_T \notin [c_T, b_T]\}}. \quad (25)$$

By Lemma 1 along with **Assumption D**, Y is a Wiener process under $\mathbb{Q}$. Then, we have by Anderson (1960) (Theorem 4.2, pp. 178-179) that Equation (25) holds. □

We provide now the proof of Theorem 2, which is based on Lemma 3.

Proof of Theorem 2. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}} \exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big] \quad &= \quad \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_{b,c}^Y \leq T\}}|W_T, v\big] \\ &\times \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big] \\ &= \Big(\sum_{j=1}^{\infty} q_{b,c}^Y(j|Y_T)\mathbf{1}_{\{Y_T \in [c_T, b_T]\}} \\ &+ \mathbf{1}_{\{Y_T \notin [c_T, b_T]\}}\Big) \quad (26) \\ &\times \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T, v\big],\end{aligned}$$

where we use Assumption (11) in the first equality, and Equation (10) from Lemma 3 along with **Assumption D** in the second equality. Finally, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|W_T,v\big] &= \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\widetilde{W}\big)|v\big] \\
&= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\big(\widetilde{W}+\int_0^T \widetilde{\theta}_s\theta_s ds\big)\big)|v\big] \\
&= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big) \\
&\quad \times \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\widetilde{\alpha}\big(\widetilde{W}+\int_0^T \widetilde{\theta}_s\theta_s ds\big)\big)\big] \\
&= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathbb{E}_{\mathbb{P}}\big[\exp\big(-\widetilde{\alpha}N\big)\big] \\
&= \exp\big(\widetilde{\alpha}\int_0^T \widetilde{\theta}_s\theta_s ds\big)\mathcal{L}_N(\widetilde{\alpha}). \qquad (27)
\end{aligned}$$

Here, we use the fact that $\widetilde{W}$ is independent from W_T in the first equality, the fact that θ_t and $\widetilde{\theta}_t$ for any $t \in [0,T]$ are $\sigma(v)$-measurable random variables in the second equality, the fact that $\mathcal{D}(\widetilde{W}+\int_0^T \widetilde{\theta}_s\theta_s ds|v)$ is standard normal under $\mathbb{Q}$ by Lemma 2 along with **Assumption D** in the third equality, the fact that $\widetilde{W}+\int_0^T \widetilde{\theta}_s\theta_s ds$ is a standard normal variable under $\mathbb{Q}$ by Lemma 2 along with **Assumption D** in the fourth equality, and Equation (2.14) in the last equality. We can deduce Equation (12) from Equations (9), (26) and (27). □

Finally, we get $P^Y_{b,c}(T)$ in the next theorem, by integrating $\mathbb{P}(\mathrm{T}^Y_{b,c} \leq T|W_T,v)$ with respect to the value of (W_T,v).

Proof of Corollary 1. We can calculate that

$$\begin{aligned}
P^Y_{b,c}(T) &= \int_{-\infty}^{\infty}\int_{\Pi_v} \mathbb{P}(\mathrm{T}^Y_{b,c} \leq T|W_T=x, v=y) \\
&\quad \times \frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)dxdP_v(y) \\
&= 1-\phi(\frac{b_T-u_T}{\sqrt{T}})+\phi(\frac{c_T-u_T}{\sqrt{T}}) \\
&\quad +\int_{c_T-u_T}^{b_T-u_T}\int_{\Pi_v} \mathbb{P}(\mathrm{T}^Y_b \leq T|W_T=x, v=y) \\
&\quad \times \frac{1}{\sqrt{2\pi T}}\exp\Big(-\frac{x^2}{2T}\Big)dxdP_v(y)
\end{aligned}$$

$$
\begin{aligned}
= \quad & 1 - \phi\Big(\frac{b - u_T}{\sqrt{T}}\Big) \\
& + \int_{-\infty}^{b-u_T} \int_{\Pi_v} \frac{1}{\sqrt{2\pi T}} \exp\Big(- \frac{x^2}{2T}\Big) \exp\big(- y_\alpha x + \frac{1}{2}\int_0^T y_{\theta,s}^2 ds\big) \\
& \times \mathbb{E}_{\mathbb{Q}}\big[\mathbf{1}_{\{\mathrm{T}_b^Y \le T\}} \exp\big(- \widetilde{\alpha}\widetilde{W}\big) | W_T = x, v = y\big] dx dP_v(y).
\end{aligned}
$$

Here, we use Equation (2), regular conditional probability and the fact that W_T and v are independent in the first equality, the fact that $\mathbb{P}(\mathrm{T}_{b,c}^Y \le T | W_T = x) = 1$ for any $x \ge b_T - u_T$ and any $x \le c_T - u_T$ in the second equality, and Equation (9) in the third equality. We have thus shown Equation (14). Equation (15) can be shown following the same first two equalities and using Equation (12) in the third equality. □

References

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