Nonparametric local estimation of the partial area under the receiver operating characteristic curve

Chang Yuan Li* and Yoann Potiron†

Abstract

We consider estimation of the receiver operating characteristic curve and the ordinal dominance curve. The nonparametric estimation is based on smoothing. We also propose estimation of the partial area under the receiver operating curve and the ordinal dominance curve. This is obtained by local estimation of the receiver operating curve and the ordinal dominance curve. We characterize feasible statistics induced by central limit theory for the estimation procedure. A numerical simulation corroborates the asymptotic theory. An empirical application to ?? shows that ??.

Keywords: receiver operating curve ; ordinal dominance curve ; partial area; nonparametric estimation; smoothing; local estimation

1 Introduction

This paper first concerns estimation of the receiver operating characteristic (ROC) curve and the ordinal dominance curve (ODC). The ROC definition and ODC definition are based on the cumulative

*Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106. E-mail: cli@pstat.ucsb.edu

†(CONTACT) Faculty of Business and Commerce, Keio University. 2-15-45 Mita, Minato-ku, Tokyo, 108-8345, Japan. E-mail: potiron@fbc.keio.ac.jp website: http://www.fbc.keio.ac.jp/~potiron

distribution function (cdf) and the quantile of two independent random variables with cdfs F and G. More specifically, we define the ROC curve at point v as

$$ROC(v) = 1 - G(F^{-1}(1-v)) \text{ for } 0 \leq v \leq 1, \tag{1}$$

and the ODC at point v as

$$ODC(v) = F(G^{-1}(v)) \text{ for } 0 \leq v \leq 1. \tag{2}$$

We propose nonparametric estimation of the two cdfs and the two quantiles based on smoothing. Then, we can estimate the ROC curve and the ODC by replacing the cdfs and quantiles with their estimates in the definitions (1) and (2). This paper also concerns estimation of the partial area under the ROC curve and the ODC. This is first obtained by local estimation of the ROC curve and the ODC. Then, we aggregate the local estimates. As far as the authors know, nonparametric estimation of the partial area under the ROC curve and the ODC based on smoothing is novel to the literature. This paper finally concerns estimation of the weighted area under the ROC curve and the ODC.

The primary application of ROC curve and ODC lies in medicine. In that field, the ROC curve and ODC are important tools describing the performance of a diagnostic test used to discriminate between healthy and diseased patients. The test scores of the patients are based on random variables measured on a continuous scale. The use of ROC curve and ODC in medicine started right after 1960. [Gonçalves et al., 2014] overview some developments on the estimation of the ROC curve. The area under the ROC curve is another important tool (see [Walter, 2005]). Since the area under the ROC curve can be limited, [Li and Fine, 2010] introduce the weighted area under the ROC curve and its application to gene selection.

There are applications or potential applications of ROC curve and ODC in almost every scientific field. [Swets and Pickett, 1982] (Appendix E) list almost 200 references in a variety of subject areas where ROC curve methods have been used. [Krzanowski and Hand, 2009] (Chapter 10) also refer substantive applications. These and later references span such diverse areas as radar signal detection, statistical quality control, psychology, polygraph lie detection, radiology, clinical chemistry, cardiology, health and behavior studies, among others. The reader is also referred to [Green et al., 1966] and [Metz, 1978] (1978) for general background.

Nonparametric approaches to estimate ROC curve, ODC, and the partial area under them have been extensively investigated. Most of them are based on Mann-Whitney U-statistics, i.e. empirical estimation. [Bamber, 1975] estimate the area under the ROC curve. [DeLong et al., 1988] compare the areas under two or more correlated ROC curves. [Wieand et al., 1989] consider the difference between ROC curves, or difference between the partial area under the ROC curves. [Hsieh and Turnbull, 1996] consider estimation of the ODC and the area under the ODC based on empirical distribution function. [Dodd and Pepe, 2003] propose semiparametric regression for the area under the ROC curve. Based on the jackknife empirical likelihood ([Jing et al., 2009]), smoothed inference procedures for ROC curve are proposed in [Gong et al., 2010] and [Adimari and Chiogna, 2012]. [Davidov and Herman, 2012] propose estimation of the ODC and the area under the ODC for stochastically ordered cdfs. New inference method of two ROC curves and ROC curves with missing data are developed in [Yang and Zhao, 2013] and [Yang and Zhao, 2015]. [Yang et al., 2017] consider partial areas under ROC curves and ODC. [Bianco et al., 2022] study robust consistent estimators for ROC curves with covariates. Simultaneous inference for partial areas under ROC curves is given in [Wechsung and Konietschke, 2023]. [Huang et al., 2023] study interval estimation for operating characteristic of continuous biomarkers with controlled sensitivity or specificity.

There are also approaches based on smoothing estimation. They are restricted to estimation of ROC and ODC, but not the area under them. [Lloyd, 1998] uses smoothed ROC curves to summarize and compare diagnostic systems. [Hall et al., 2004] gives confidence intervals for receiver operating characteristic curves. [Peng and Zhou, 2004] consider local linear smoothing of receiver operating characteristic (ROC) curves.

Although smoothing estimation for ROC curves is generally better than empirical estimation for ROC curves (see [Lloyd and Yong, 1999]), there is as far as the authors know no method for estimation of the partial area under the ROC curve or ODC based on smoothing estimation. In this paper, we propose to fill this gap with the use of delta-sequences, i.e. a sequence of functions approaching the Dirac function asymptotically. They are considered since they are general. They cover several types of smoothing estimation including the kernel estimation, orthogonal series estimation, Fourier transform estimation, and the histogram (see [Walter and Blum, 1979]). They are also suitable since the cdf estimator is proven to be uniformly convergent in [Winter, 1973], and satisfying Chung-Smirnov property in [Winter, 1979] for the smoothing estimator case. [Reiss, 1981] shows that the empirical estimator is asymptotically deficient compared to the smoothing estimator. These delta-sequences were introduced in [Rosenblatt, 1956] (see Section 4), and also used in [Watson and Leadbetter, 1964a] (see Equations (2.1) and (2.2), p. 176).

Theorem 1 gives the central limit theory (CLT) for estimation based on delta-sequences of the ROC curve and the ODC. Theorem 2 provides the CLT for local estimation of the partial area under the ROC curve and the ODC. Theorem 3 provides the CLT for local estimation of the weighted area under the ROC curve and the ODC. They also provide feasible statistics induced by the CLT. All these results are novel to the literature.

The main argument in the proofs is the use of the convergence in distribution in the sense of stochastic processes. By stochastic processes, we mean that the convergence is obtained jointly for any point between the starting point and the ending point of the area. More specifically, we show the convergence in distribution of the process of ROC curve statistics to a limit which is a Gaussian process.

2 Setting

In this section, we introduce the setting, the definition of the partial area under the ROC curve and the ODC, and the definition of the weighted area under the ROC curve and the ODC.

We observe n independent random variables $X_1, \cdots, X_n$ on the real positive numbers $\mathbb{R}^+$ with cdf defined as F and probability distribution function (pdf) denoted by f. We also observe m independent random variables $Y_1, \cdots, Y_m$ on the real positive numbers $\mathbb{R}^+$ with cdf defined as G and pdf denoted by g. We assume that the random variables $X_1, \cdots, X_n$ are independent from the random variables $Y_1, \cdots, Y_m$. When used for a diagnostic test, the random variables $X_1, \cdots, X_n$ are the test scores of healthy patients and the random variables $Y_1, \cdots, Y_m$ are the test scores of diseased patients. The test scores of the patients are based on random variables measured on a continuous scale.

The partial area under the ROC curve or ODC is another important tool (see [Walter, 2005]). We define the partial area under the ROC curve from the starting point v_0 to the final point v as

$$pROC(v_0, v) = \int_{v_0}^{v} ROC(u)du \text{ for } 0 \leq v_0 < v \leq 1. \tag{3}$$

We define the partial area under the ODC curve from the starting point v_0 to the final point v as

$$pODC(v_0, v) = \int_{v_0}^{v} ODC(u)du \text{ for } 0 \leq v_0 < v \leq 1. \tag{4}$$

Since the area under the ROC curve and ODC can be limited, [Li and Fine, 2010] introduce the weighted area under the ROC curve and its application to gene selection. Thus, we introduce h_R as a known pdf on the interval $[0, 1]$. We define the weighted area under the ROC curve from the starting point v_0 to the final point v as

$$wROC(v_0, v) = \int_{v_0}^{v} ROC(u)h_R(u)du \text{ for } 0 \leq v_0 < v \leq 1. \tag{5}$$

Finally, we introduce h_0 as a known pdf on the interval $[0, 1]$. We define the weighted area under the ODC curve from the starting point v_0 to the final point v as

$$wODC(v_0, v) = \int_{v_0}^{v} ODC(u)h_0(u)du \text{ for } 0 \leq v_0 < v \leq 1. \tag{6}$$

3 Estimation

In this section, we introduce nonparametric estimation based on delta-sequences for ROC curve and ODC. We also introduce local estimation for the partial area under the ROC curve and the partial area under the ODC. Finally, we introduce local estimation for the weighted area under the ROC curve and the weighted area under the ODC.

We prefer most of the time not to write explicitly the dependence on n in definitions. We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F})$, where $\mathcal{F}$ is a σ-field. We consider nonparametric estimation of the cdf, based on delta-sequences δ.

Following Section 4 of [Watson and Leadbetter, 1964b], we can then estimate the pdfs and cdfs as

$$\widehat{f}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta(t - X_i) \ , \ \widehat{g}(t) = \frac{1}{n}\sum_{j=1}^{m} \delta(t - Y_j) \text{ for any } t \geq 0, \tag{7}$$

$$\widehat{F}(t) = \int_0^t \widehat{f}(u)du \ , \ \widehat{G}(t) = \int_0^t \widehat{g}(u)du \text{ for any } t \geq 0. \tag{8}$$

We define the empirical estimator of the quantiles as

$$\widehat{F}^{-1}(v) = \inf\{t \in \mathbb{R}^+ \text{ s.t. } \widehat{F}(t) = v\} \text{ and } \widehat{G}^{-1}(v) = \inf\{t \in \mathbb{R}^+ \text{ s.t. } \widehat{G}(t) = v\}. \tag{9}$$

We define the estimator of ROC curve as

$$\widehat{ROC}(v) = 1 - \widehat{G}(\widehat{F}^{-1}(1 - v)), \tag{10}$$

and the estimator of ODC curve as

$$\widehat{ODC}(v) = \widehat{F}(\widehat{G}^{-1}(v)). \tag{11}$$

We define the asymptotic variance for estimation of ROC curve as

$$V_R(v) = \lambda G(F^{-1}(1 - v))(1 - G(F^{-1}(1 - v)) + \frac{v(1 - v)g(F^{-1}(1 - v))^2}{f(F^{-1}(1 - v))^2} \tag{12}$$

We also define the asymptotic variance for estimation of ODC as

$$V_O(v) = F(G^{-1}(v))(1 - F(G^{-1}(v)) + \frac{\lambda v(1 - v)f(G^{-1}(v))^2}{g(G^{-1}(v))^2} \tag{13}$$

We define the estimator of asymptotic variance for ROC curve estimation as

$$\widehat{V}_R(v) = \lambda \widehat{G}(\widehat{F}^{-1}(1 - v))(1 - \widehat{G}(F^{-1}(1 - v)) + \frac{v(1 - v)\widehat{g}(\widehat{F}^{-1}(1 - v))^2}{\widehat{f}(\widehat{F}^{-1}(1 - v))^2} \tag{14}$$

We also define the estimator of asymptotic variance for ODC estimation as

$$\widehat{V}_O(v) = \widehat{F}(\widehat{G}^{-1}(v))(1 - \widehat{F}(\widehat{G}^{-1}(v)) + \frac{\lambda v(1 - v)\widehat{f}(\widehat{G}^{-1}(v))^2}{\widehat{g}(\widehat{G}^{-1}(v))^2}. \tag{15}$$

To estimate the partial area under the ROC curve (3), we consider local estimation

$$\widehat{pROC}(v_0, v) = \sum_{k=0}^{M-1} \widehat{ROC}(v_k)\Delta. \tag{16}$$

Here, M is the number of points of approximation, $\Delta = \frac{v-v_0}{M}$ is the length between each point of approximation, and the points of approximation are $v_k = v_0 + k\Delta$ for $k = 0, \cdots, M-1$. To estimate the partial area under the ODC curve (4), we also consider local estimation

$$\widehat{pODC}(v_0, v) = \sum_{k=0}^{M-1} \widehat{ODC}(v_k)\Delta. \tag{17}$$

We define the asymptotic variance for estimation of the partial area under the ROC curve as

$$V_{pR}(v_0, v) = \int_{v_0}^{v} (1 + 2(v-u))V_R(u)du. \tag{18}$$

We also define the asymptotic variance for estimation of the partial area under the ODC as

$$V_{pO}(v_0, v) = \int_{v_0}^{v} (1 + 2(v-u))V_O(u)du. \tag{19}$$

We define the estimator of asymptotic variance for the partial area under the ROC curve as

$$\widehat{V}_{pR}(v_0, v) = \sum_{k=0}^{M-1} (1 + 2(v-v_k))\widehat{V}_R(v_k)\Delta. \tag{20}$$

We also define the estimator of asymptotic variance for the partial area under the ODC curve as

$$\widehat{V}_{pO}(v_0, v) = \sum_{k=0}^{M-1} (1 + 2(v-v_k))\widehat{V}_O(v_k)\Delta. \tag{21}$$

To estimate the weighted area under the ROC curve (5), we also consider local estimation

$$\widehat{wROC}(v_0, v) = \sum_{k=0}^{M-1} \widehat{ROC}(v_k)h_R(v_k)\Delta. \tag{22}$$

To estimate the weighted area under the ODC curve (6), we also consider local estimation

$$\widehat{wODC}(v_0, v) = \sum_{k=0}^{M-1} \widehat{ODC}(v_k)h_0(v_k)\Delta. \tag{23}$$

We define the asymptotic variance for estimation of the weighted area under the ROC curve as

$$V_{wR}(v_0, v) = \int_{v_0}^{v} (1 + 2(v-u))V_R(u)h_R^2(u)du. \tag{24}$$

We also define the asymptotic variance for estimation of the weighted area under the ODC as

$$V_{wO}(v_0, v) = \int_{v_0}^{v} (1 + 2(v - u))V_O(u)h_O^2(u)du. \tag{25}$$

We define the estimator of asymptotic variance for the weighted area under the ROC curve as

$$\widehat{V}_{wR}(v_0, v) = \sum_{k=0}^{M-1} \widehat{V}_R(v_k)h(v_k)\Delta. \tag{26}$$

Finally, we define the estimator of asymptotic variance for the weighted area under the ODC as

$$\widehat{V}_{wO}(v_0, v) = \sum_{k=0}^{M-1} \widehat{V}_O(v_k)h(v_k)\Delta. \tag{27}$$

4 Theory

We introduce a sequence of positive functions $\delta : \mathbb{R} \to \mathbb{R}^+$ which will be called positive delta-sequence if it satisfies the following set of conditions.

Condition 1. (a) We have $\int_{\mathbb{R}} \delta(t)dt = 1$ for any $n \in \mathbb{N}$.

(b) We have $\sup_{|t|\geq\kappa} \delta(t) \to 0$ as $n \to \infty$ for any $\kappa > 0$.

(c) We have $\int_{|t|\geq\kappa} \delta(t)dt \to 0$ as $n \to \infty$ for any $\kappa > 0$.

For example, we can consider the kernel

$$\delta(t) = \frac{r(t/A)}{A\int_{\mathbb{R}} r(t)dt}.$$

Here, we have that $r : \mathbb{R} \to \mathbb{R}^+$ is a bounded integrable function satisfying $tr(t) \to 0$ as $\mid t \mid \to \infty$, and A is the bandwidth satisfying $A \to 0$ as $n \to \infty$. When compared to the general definition of delta-sequences given in [Watson and Leadbetter, 1964b] (Section 2, pp. 102-103), we restrict to the positive case since we want to avoid estimated negative pdf and cdf which would imply that estimator of the quantiles (9) is not well-defined. This comes with a price of not allowing for kernel improving the rate of convergence of the bias and the MSE (see [Singpurwalla and Wong, 1983]).

Moreover, we introduce the squared-average of the delta-sequence δ as

$$\alpha = \int_{-\infty}^{\infty} \delta^2(t)dt.$$

We also introduce a real positive number $\mu > 0$. Finally, we introduce the $(2+\mu)$ power-average of the delta-sequence δ as

$$\gamma = \int_{-\infty}^{\infty} \delta^{2+\mu}(t)dt.$$

Let us introduce a set of conditions required to derive the CLT tor estimation of ROC curve and ODC.

Condition 2. (a) We have $\alpha < \infty$, $\gamma < \infty$ for any $n \in \mathbb{N}^*$, $\frac{\alpha}{n} \to 0$ and $\frac{\gamma}{n^{\mu/2}\alpha^{1+\mu/2}} \to 0$ as $n \to \infty$.

(b) There exists a real positive number $\lambda > 0$ such that $\frac{n}{m} \to \lambda$.

(c) The random variables $X_1, \cdots, X_n$ are independent from the random variables $Y_1, \cdots, Y_m$.

(d) The cdfs F and G are strictly increasing on $\mathbb{R}^+$.

(e) The cdfs F and G are continuously differentiable on $\mathbb{R}^+$.

(f) We have the rescaled bias of estimation for F that converges to 0, i.e. $\sqrt{n}(\mathbb{E}[\widehat{F}(G^{-1}(v))] - F(G^{-1}(v))) \to 0$ and $\sqrt{n}(\mathbb{E}[\widehat{F}(F^{-1}(1-v))] - F(F^{-1}(1-v)) \to 0$ as $n \to \infty$. We also have the rescaled bias of estimation for G that converges to 0, i.e. $\sqrt{m}(\mathbb{E}[\widehat{G}(F^{-1}(1-v))] - G(F^{-1}(1-v))) \to 0$ and $\sqrt{m}(\mathbb{E}[\widehat{G}(G^{-1}(v))] - G(G^{-1}(v))) \to 0$ as $m \to \infty$.

Condition 2 (a) is required to apply Theorem 6 (p. 112) in [Watson and Leadbetter, 1964b]. Conditions 2 (b) and (c) are usual for estimation of ROC curve and ODC, and appears in Section 2 (p. 28) of [Hsieh and Turnbull, 1996]. Condition 2 (d) is required to obtain unicity of the quantiles. Conditions 2 (e) and (f) correspond to Conditions 2) and 3) in [Nadaraya, 1964] (p. 499). Our assumptions are weaker, since we do not require Condition 1) from [Nadaraya, 1964] (p. 499). This is due to the fact that we use in the proofs the uniform consistency of the pdf estimator on a compact space, rather than on the real positive space $\mathbb{R}^+$.

We define N as a random variable with standard normal distribution. We first state the CLT for estimation of ROC curve and ODC in the following theorem. It also provides feasible statistics induced by the CLT.

Theorem 1. *Under Conditions 1 and 2, we obtain the CLT for estimation of ROC curve and ODC curve,*

$$\sqrt{n}\frac{\widehat{ROC}(v) - ROC(v)}{\sqrt{V_R(v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v < 1, \tag{28}$$

$$\sqrt{n}\frac{\widehat{ODC}(v) - ODC(v)}{\sqrt{V_O(v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v < 1, \tag{29}$$

as $n \to \infty$. We also obtain the feasible CLT for estimation of ROC curve and ODC curve,

$$\sqrt{n}\frac{\widehat{ROC}(v) - ROC(v)}{\sqrt{\widehat{V}_R(v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v < 1, \tag{30}$$

$$\sqrt{n}\frac{\widehat{ODC}(v) - ODC(v)}{\sqrt{\widehat{V}_O(v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v < 1, \tag{31}$$

as $n \to \infty$.

For two real numbers a and b, we denote by $a \vee b$ the minimum of a and b and by $a \wedge b$ the maximum of a and b. To obtain the CLT for estimation of the partial area under the ROC curve and the partial area under the ODC curve, we make another set of conditions.

Condition 3. (a) We have $\Delta \to 0$ and such that $\frac{\sqrt{n}}{\Delta} \to 0$ as $n \to \infty$.

(b) We have the pdfs f and g are bounded uniformly by a constant strictly bigger than 0, i.e. there exists $K > 0$ such that

$$\inf_{t \in [F^{-1}(1-v), F^{-1}(1-v_0)]} f(t) \geq K \text{ and } \inf_{t \in [G^{-1}(v_0), G^{-1}(v)]} g(t) \geq K.$$

(c) We have the rescaled bias of estimation for F that converges to 0 uniformly, i.e.

$$\sup_{t \in [G^{-1}(v_0) \vee F^{-1}(1-v), G^{-1}(v) \wedge F^{-1}(1-v_0)]} \sqrt{n}(\mathbb{E}[\widehat{F}(t)] - F(t)) \to 0$$

as $n \to \infty$. We also have the bias of estimation for G that converges to 0 uniformly, i.e.

$$\sup_{t \in [G^{-1}(v_0) \vee F^{-1}(1-v), G^{-1}(v) \wedge F^{-1}(1-v_0)]} \sqrt{m}(\mathbb{E}[\widehat{G}(t)] - G(t)) \to 0,$$

as $m \to \infty$.

Condition 3 (a) is required for local estimation. Condition 3 (b) is used to prove the uniform consistency for quantile estimation.

We now state the CLT for estimation of the partial area under the ROC curve and the partial area under the ODC in the following theorem. It also provides feasible statistics induced by the CLT.

Theorem 2. *We assume that Conditions 1, 2 and 3 hold. We obtain the CLT for estimation of the partial area under the ROC curve and the partial area under the ODC curve,*

$$\sqrt{n}\frac{\widehat{pROC}(v_0, v) - pROC(v_0, v)}{\sqrt{V_{pR}(v_0, v)}} \overset{\mathcal{D}}{\to} N \text{ for any } 0 < v_0 < v < 1, \tag{32}$$

$$\sqrt{n}\frac{\widehat{pODC}(v_0, v) - pODC(v_0, v)}{\sqrt{V_{pO}(v_0, v)}} \overset{\mathcal{D}}{\to} N \text{ for any } 0 < v_0 < v < 1, \tag{33}$$

as $n \to \infty$. We also obtain the feasible CLT for estimation of the partial area under the ROC curve and the partial area under the ODC curve,

$$\sqrt{n}\frac{\widehat{pROC}(v_0, v) - pROC(v_0, v)}{\sqrt{\widehat{V}_{pR}(v_0, v)}} \overset{\mathcal{D}}{\to} N \text{ for any } 0 < v_0 < v < 1, \tag{34}$$

$$\sqrt{n}\frac{\widehat{pODC}(v_0, v) - pODC(v_0, v)}{\sqrt{\widehat{V}_{pO}(v_0, v)}} \overset{\mathcal{D}}{\to} N \text{ for any } 0 < v_0 < v < 1, \tag{35}$$

as $n \to \infty$.

Finally, we state the CLT for estimation of the weighted area under the ROC curve and the weighted area under the ODC curve in the following theorem. It also provides feasible statistics induced by the CLT.

Theorem 3. *We assume that Conditions 1, 2 and 3 hold. We obtain the CLT for estimation of the* *weighted area under the ROC curve and the weighted area under the ODC curve*

$$\sqrt{n}\frac{\widehat{wROC}(v_0,v) - wROC(v_0,v)}{\sqrt{V_{wR}(v_0,v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v_0 < v < 1, \tag{36}$$

$$\sqrt{n}\frac{\widehat{wODC}(v_0,v) - wODC(v_0,v)}{\sqrt{V_{wO}(v_0,v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v_0 < v < 1, \tag{37}$$

as $n \to \infty$. *We also obtain the feasible CLT for estimation of the weighted area under the ROC curve and the weighted area under the ODC curve*

$$\sqrt{n}\frac{\widehat{wROC}(v_0,v) - wROC(v_0,v)}{\sqrt{\widehat{V}_{wR}(v_0,v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v_0 < v < 1, \tag{38}$$

$$\sqrt{n}\frac{\widehat{wODC}(v_0,v) - wODC(v_0,v)}{\sqrt{\widehat{V}_{wO}(v_0,v)}} \xrightarrow{\mathcal{D}} N \text{ for any } 0 < v_0 < v < 1, \tag{39}$$

as $n \to \infty$.

5 Numerical study

In this section, we first report how the feasible statistics induced by CLT from Section 4 behave in finite sample. We also compare these estimators to three concurrent estimators from the literature ([Yang et al., 2017]). The estimators are normal approximation (NA) from Corollary 1, jacknife estimation (JKN) from Theorem 1, and jackknife empirical likelihood (JEL) from Theorem 2.

The setting will be: two different models, five different (n,m) values, six different estimators (ROC, ODC, pROC, pODV, wROC, wODC). There will be six tables (see Table 1), one for each estimation. First, focus on filling in Table 1

We first introduce the setting.

A. We observe independent random variables $X_1,\cdots,X_m \sim F, Y_1,\cdots,Y_n \sim G$.

B. $ROC(t) = 1 - G(F^{-1}(1-t)), ODC(t) = F(G^{-1}(t))$.

C. Define $iROC(t) = \int_0^t ROC(u)du, iODC(t) = \int_0^t ODC(u)du$, we have $iROC(1) = iODC(1) = \mathbb{P}(X < Y)$.

Two Estimators

A. Based on plugging in empirical distribution function (EDF) $\tilde{F}(t) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{(X_i \le t)}, \tilde{G}(t) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{(Y_j \le t)}$

$$\widetilde{iODC}(v) = \int_0^v \tilde{F}(\tilde{G}^{-1}(t))dt = \int_{-\infty}^{\tilde{G}^{-1}(v)} \tilde{F}(u)d\tilde{G}(u) = \frac{1}{mn}\sum_{i,j} \mathbb{1}_{(X_i<Y_j)}\mathbb{1}_{(Y_j \le \tilde{G}^{-1}(v))} \tag{40}$$

$$\widetilde{iROC}(v) = \frac{1}{mn}\sum_{i,j} \mathbb{1}_{(X_i<Y_j)}\mathbb{1}_{(\tilde{F}(X_i)\ge 1-v)} \tag{41}$$

where $v \in (0, 1]$.

B. Based on kernel density estimation $\hat{F}(t), \hat{G}(t)$ and numerical integration (trapezoidal rule). Let $\{t_k\}_{k=1}^{K}$ partition $[0, v]$, i.e. $0 = t_0 < t_1 < \cdots < t_K = v$. $\Delta t_k = t_k - t_{k-1}$.

$$\widehat{iODC}(v) = \sum_{k=1}^{K} \Delta t_k \cdot \tfrac{\hat{F}(\hat{G}^{-1}(t_k)) + \hat{F}(\hat{G}^{-1}(t_{k-1}))}{2} \approx \int_0^v \hat{F}(\hat{G}^{-1}(t))dt \tag{42}$$

$$\widehat{iROC}(v) = \sum_{k=1}^{K} \Delta t_k \cdot (1 - \tfrac{\hat{G}(\hat{F}^{-1}(1-t_k)) + \hat{G}(\hat{F}^{-1}(1-t_{k-1}))}{2}) \approx \int_0^v 1 - \hat{G}(\hat{F}^{-1}(1-t))dt \tag{43}$$

6 Empirical application

7 Discussion

8 Conclusion

In this paper,

The code is available online at ???

Supplementary materials

All proofs of the theory can be found in the supplementary materials.

Table 1: Summary statistics for ROC curve with feasible statistics. The number of replications is ??.

Method			Ours		Alternative 1		Alternative 2		Alternative 3	
Model	n	m	Mean	Variance	Mean	Var	Mean	Var	Mean	Var
1	20	30								
1	40	70								
1	70	120								
1	100	150								
1	200	200								
2	20	30								
2	40	70								
2	70	120								
2	100	150								
2	200	200								

Funding

Financial support from the Japanese Society for the Promotion of Science under grant 23H00807 (Potiron) is gratefully acknowledged.

References

[Adimari and Chiogna, 2012] Adimari, G. and Chiogna, M. (2012). Jackknife empirical likelihood based confidence intervals for partial areas under roc curves. *Statistica Sinica*, pages 1457–1477.

[Bamber, 1975] Bamber, D. (1975). The area above the ordinal dominance graph and the area below the receiver operating characteristic graph. *Journal of mathematical psychology*, 12(4):387–415.

[Bianco et al., 2022] Bianco, A. M., Boente, G., and González-Manteiga, W. (2022). Robust consistent estimators for roc curves with covariates. *Electronic Journal of Statistics*, 16(2):4133–4161.

[Davidov and Herman, 2012] Davidov, O. and Herman, A. (2012). Ordinal dominance curve based inference for stochastically ordered distributions. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 74(5):825–847.

[DeLong et al., 1988] DeLong, E. R., DeLong, D. M., and Clarke-Pearson, D. L. (1988). Comparing the areas under two or more correlated receiver operating characteristic curves: a nonparametric approach. *Biometrics*, pages 837–845.

[Dodd and Pepe, 2003] Dodd, L. E. and Pepe, M. S. (2003). Semiparametric regression for the area under the receiver operating characteristic curve. *Journal of the American Statistical Association*, 98(462):409–417.

[Gonçalves et al., 2014] Gonçalves, L., Subtil, A., Oliveira, M. R., and de Zea Bermudez, P. (2014). Roc curve estimation: An overview. *REVSTAT-Statistical journal*, 12(1):1–20.

[Gong et al., 2010] Gong, Y., Peng, L., and Qi, Y. (2010). Smoothed jackknife empirical likelihood method for roc curve. *Journal of Multivariate Analysis*, 101(6):1520–1531.

[Green et al., 1966] Green, D. M., Swets, J. A., et al. (1966). *Signal detection theory and psychophysics*, volume 1. Wiley New York.

[Hall et al., 2004] Hall, P., Hyndman, R. J., and Fan, Y. (2004). Nonparametric confidence intervals for receiver operating characteristic curves. *Biometrika*, 91(3):743–750.

[Hsieh and Turnbull, 1996] Hsieh, F. and Turnbull, B. W. (1996). Nonparametric and semiparametric estimation of the receiver operating characteristic curve. *The Annals of Statistics*, 24(1):25–40.

[Huang et al., 2023] Huang, Y., Parakati, I., Patil, D. H., and Sanda, M. G. (2023). Interval estimation for operating characteristic of continuous biomarkers with controlled sensitivity or specificity. *Statistica Sinica*, 33(1):193.

[Jing et al., 2009] Jing, B.-Y., Yuan, J., and Zhou, W. (2009). Jackknife empirical likelihood. *Journal of the American Statistical Association*, 104(487):1224–1232.

[Kikuchi et al., 2024] Kikuchi, J. K., Li, C. Y., and Potiron, Y. (2024). Nonparametric estimation of hitting-time variance. *Working paper available at https://www.fbc.keio.ac.jp/∼potiron/KikuchiLiPotironworkingpaper.pdf*.

[Krzanowski and Hand, 2009] Krzanowski, W. J. and Hand, D. J. (2009). *ROC curves for continuous data*. Chapman and Hall/CRC.

[Li and Fine, 2010] Li, J. and Fine, J. P. (2010). Weighted area under the receiver operating characteristic curve and its application to gene selection. *Journal of the Royal Statistical Society Series C: Applied Statistics*, 59(4):673–692.

[Lloyd, 1998] Lloyd, C. J. (1998). Using smoothed receiver operating characteristic curves to summarize and compare diagnostic systems. *Journal of the American Statistical Association*, 93(444):1356–1364.

[Lloyd and Yong, 1999] Lloyd, C. J. and Yong, Z. (1999). Kernel estimators of the roc curve are better than empirical. *Statistics & Probability Letters*, 44(3):221–228.

[Loeve, 1977] Loeve, M. M. (1977). *Probability Theory I.* Springer, 4th edition.

[Metz, 1978] Metz, C. E. (1978). Basic principles of ROC analysis. In *Seminars in nuclear medicine*, volume 8, pages 283–298. Elsevier.

[Nadaraya, 1964] Nadaraya, E. A. (1964). Some new estimates for distribution functions. *Theory of Probability & Its Applications*, 9(3):497–500.

[Peng and Zhou, 2004] Peng, L. and Zhou, X.-H. (2004). Local linear smoothing of receiver operating characteristic (roc) curves. *Journal of Statistical Planning and Inference*, 118(1-2):129–143.

[Reiss, 1981] Reiss, R.-D. (1981). Nonparametric estimation of smooth distribution functions. *Scandinavian Journal of Statistics*, 8(2):116–119.

[Rosenblatt, 1956] Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function. *The Annals of Mathematical Statistics*, pages 832–837.

[Singpurwalla and Wong, 1983] Singpurwalla, N. D. and Wong, M.-Y. (1983). Kernel estimators of the failure-rate function and density estimation: an analogy. *Journal of the American Statistical Association*, 78(382):478–481.

[Swets and Pickett, 1982] Swets, J. and Pickett, R. M. (1982). *Evaluation of Diagnostic Systems: Methods from Signal Detection Theory.* Academic Press, New York.

[Walter and Blum, 1979] Walter, G. and Blum, J. (1979). Probability density estimation using delta sequences. *The Annals of Statistics*, pages 328–340.

[Walter, 2005] Walter, S. D. (2005). The partial area under the summary roc curve. *Statistics in medicine*, 24(13):2025–2040.

[Watson and Leadbetter, 1964a] Watson, G. and Leadbetter, M. (1964a). Hazard analysis. i. *Biometrika*, 51(1):175–184.

[Watson and Leadbetter, 1964b] Watson, G. S. and Leadbetter, M. (1964b). Hazard analysis ii. *Sankhyā: The Indian Journal of Statistics, Series A*, 26(1):101–116.

[Wechsung and Konietschke, 2023] Wechsung, M. and Konietschke, F. (2023). Simultaneous inference for partial areas under receiver operating curves—with a view towards efficiency. *Journal of Statistical Planning and Inference*, 226:71–85.

[Wieand et al., 1989] Wieand, S., Gail, M. H., James, B. R., and James, K. L. (1989). A family of nonparametric statistics for comparing diagnostic markers with paired or unpaired data. *Biometrika*, 76(3):585–592.

[Winter, 1973] Winter, B. (1973). Strong uniform consistency of integrals of density estimators. *Canadian Journal of Statistics*, 1(1-2):247–253.

[Winter, 1979] Winter, B. (1979). Convergence rate of perturbed empirical distribution functions. *Journal of Applied Probability*, 16(1):163–173.

[Yang et al., 2017] Yang, H., Lu, K., and Zhao, Y. (2017). A nonparametric approach for partial areas under roc curves and ordinal dominance curves. *Statistica Sinica*, pages 357–371.

[Yang and Zhao, 2013] Yang, H. and Zhao, Y. (2013). Smoothed jackknife empirical likelihood inference for the difference of roc curves. *Journal of Multivariate Analysis*, 115:270–284.

[Yang and Zhao, 2015] Yang, H. and Zhao, Y. (2015). Smoothed jackknife empirical likelihood inference for roc curves with missing data. *Journal of Multivariate Analysis*, 140:123–138.

Supplementary materials

This part corresponds to the supplementary materials of "Nonparametric local estimation of the partial area under the receiver operating characteristic curve" by Chang Yuan Li and Yoann Potiron submitted to the Journal of American Statistical Association. All the proofs of the theory can be found in Section 9.

9 Proofs

We start with the CLT for estimation of cdfs. The following lemma is an application of Theorem 6 (p. 112) in [Watson and Leadbetter, 1964b].

Lemma 1. *We assume that Condition 1 and Conditions 2 (a), (b) and (f) hold. We have the CLT for the cdf estimators*

$$\sqrt{n}(\widehat{F}(t) - F(t)) \xrightarrow{\mathcal{D}} N_F(t) \text{ for any } t > 0 \text{ such that } 0 < F(t) < 1, \tag{44}$$

$$\sqrt{n}(\widehat{G}(t) - G(t)) \xrightarrow{\mathcal{D}} N_G(t) \text{ for any } t > 0 \text{ such that } 0 < G(t) < 1, \tag{45}$$

as $n \to \infty$. Here, $N_F(t)$ is a random variable with normal distribution characterized by its mean equal to zero and its variance equal to $F(t)(1 - F(t))$. Also, $N_G(t)$ is a random variable with normal distribution characterized by its mean equal to zero and its variance equal to $\lambda G(t)(1 - G(t))$.

Proof of Lemma 1. We have that

$$\sqrt{n}(\widehat{F}(t) - F(t)) = \sqrt{n}(\widehat{F}(t) - \mathbb{E}[\widehat{F}(t)]) + \sqrt{n}(\mathbb{E}[\widehat{F}(t)] - F(t)).$$

By Condition 2 (f), we can deduce that the random variables

$$\sqrt{n}(\widehat{F}(t) - F(t))$$

and

$$\sqrt{n}(\widehat{F}(t) - \mathbb{E}[\widehat{F}(t)])$$

converge to the same limit in distribution. Then, an application of [Watson and Leadbetter, 1964b] (Theorem 6 (iii)) with Conditions 1 and 2 (a) gives Expression (44).

We have that

$$\sqrt{n}(\widehat{G}(t) - G(t)) = \sqrt{m}\frac{\sqrt{n}}{\sqrt{m}}(\widehat{G}(t) - G(t)).$$

By Slutsky's theorem and Condition 2 (b), we can deduce that the random variables

$$\sqrt{n}(\widehat{G}(t) - G(t))$$

and

$$\sqrt{m}\sqrt{\lambda}(\widehat{G}(t) - G(t))$$

converge to the same limit in distribution. Then, we have that

$$\sqrt{m}\sqrt{\lambda}(\widehat{G}(t) - G(t)) = \sqrt{m}\sqrt{\lambda}(\widehat{G}(t) - \mathbb{E}[\widehat{G}(t)]) + \sqrt{m}\sqrt{\lambda}(\mathbb{E}[\widehat{G}(t)] - G(t)).$$

By Condition 2 (f), we can deduce that

$$\sqrt{m}\sqrt{\lambda}(\widehat{G}(t) - G(t))$$

and

$$\sqrt{m}\sqrt{\lambda}(\widehat{G}(t) - \mathbb{E}[\widehat{G}(t)])$$

converge to the same limit in distribution. Then, an application of [Watson and Leadbetter, 1964b] (Theorem 6 (iii)) with Conditions 1 and 2 (a) gives Expression (45). □

We give now the uniform consistency of the cdf estimators on the space of real positive numbers. The following lemma is an application of Theorem 1 (p. 248) in [Winter, 1973].

Lemma 2. *Under Conditions 1 and 2 (b), we have the uniform consistency of the cdf estimators*

$$\sup_{t\in\mathbb{R}^+} \left|\widehat{F}(t) - F(t)\right| \xrightarrow{\mathbb{P}} 0 \text{ and } \sup_{t\in\mathbb{R}^+} \left|\widehat{G}(t) - G(t)\right| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$.

Proof of Lemma 2. This is a direct application of Theorem 1 (p. 248) in [Winter, 1973] and Condition 2 (b). □

We give now the uniform consistency of the pdf estimators on any bounded interval of $\mathbb{R}^+$. The following lemma extends Lemmas 6 and 7 in [Kikuchi et al., 2024].

Lemma 3. *We assume that Conditions 1, 2 (b) and (e) hold. We have the uniform consistency of the pdf estimators on any interval* $[a, b]$ *of* $\mathbb{R}^+$ *such that* $0 < a < b$*, i.e.*

$$\sup_{t\in[a,b]} \left|\widehat{f}(t) - f(t)\right| \xrightarrow{\mathbb{P}} 0 \text{ and } \sup_{t\in[a,b]} \left|\widehat{g}(t) - g(t)\right| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$.

Proof of Lemma 3. As $\mathbb{L}^2$-convergence implies $\mathbb{P}$-convergence, it is sufficient to show the $\mathbb{L}^2$-convergence. We have that

$$\mathbb{E}[(\widehat{f}(t) - f(t))^2] = (\mathbb{E}[\widehat{f}(t)] - f(t))^2 + \mathrm{Var}[\widehat{f}(t)].$$

Thus, we can deduce that

$$\sup_{t\in[a,b]} \mathbb{E}[(\widehat{f}(t) - f(t))^2] \leq \sup_{t\in[a,b]} (\mathbb{E}[\widehat{f}(t)] - f(t))^2 + \sup_{t\in[a,b]} \mathrm{Var}[\widehat{f}(t)].$$

Since

$$\mathbb{E}[\widehat{f}(t)] = \int_{\mathbb{R}^+} \delta_n(s - t) f(s) ds,$$

we can apply Lemma 6 in [Kikuchi et al., 2024] with Conditions 1 and 2 (e) to obtain

$$\sup_{t\in[a,b]} (\mathbb{E}[\widehat{f}(t)] - f(t))^2 \to 0.$$

We can also apply Lemma 7 in [Kikuchi et al., 2024] with Conditions 1 and 2 (e) to obtain

$$\sup_{t\in[a,b]} \mathrm{Var}[\widehat{f}(t)] \to 0.$$

The same arguments with Condition 2 (b) yield

$$\sup_{t\in[a,b]} \left|\widehat{g}(t) - g(t)\right| \xrightarrow{\mathbb{P}} 0,$$

□

We give now the consistency of the quantile estimators. The following lemma extends [Nadaraya, 1964] (Section 3) to the case of delta-sequences.

Lemma 4. *We assume that Conditions 1, 2 (b) and (d) hold. We have the consistency of the quantile estimators*

$$\left|\widehat{F}^{-1}(v) - F^{-1}(v)\right| \xrightarrow{\mathbb{P}} 0 \text{ and } \left|\widehat{G}^{-1}(v) - G^{-1}(v)\right| \xrightarrow{\mathbb{P}} 0 \text{ for any } v \in (0,1), \tag{46}$$

as $n \to \infty$.

Proof of Lemma 4. For $\epsilon > 0$, we consider

$$\delta = \min(F(F^{-1}(v) + \epsilon) - v, v - F(F^{-1}(v) - \epsilon)).$$

By Condition 2 (d), we can deduce that $\delta > 0$. By definition of δ, we obtain

$$\mathbb{P}\left(\left|\widehat{F}^{-1}(v) - F^{-1}(v)\right| > \epsilon\right) \leq \mathbb{P}\left(\left|F(\widehat{F}^{-1}(v)) - v\right| > \delta\right). \tag{47}$$

By definition of the quantile estimator (9), we obtain

$$\left|F(\widehat{F}^{-1}(v)) - v\right| = \left|F(\widehat{F}^{-1}(v)) - \widehat{F}(\widehat{F}^{-1}(v))\right|. \tag{48}$$

By bounding the term in the right hand-side of Equation (48), we obtain

$$\left|F(\widehat{F}^{-1}(v)) - \widehat{F}(\widehat{F}^{-1}(v))\right| \leq \sup_{t \in \mathbb{R}^+} \left|F(t) - \widehat{F}(t)\right|. \tag{49}$$

By Lemma 2 with Expressions (47), (48) and (49), we can deduce that

$$\left|\widehat{F}^{-1}(v) - F^{-1}(v)\right| \xrightarrow{\mathbb{P}} 0.$$

The same arguments with Condition 2 (b) yield

$$\left|\widehat{G}^{-1}(v) - G^{-1}(v)\right| \xrightarrow{\mathbb{P}} 0.$$

□

We continue with the CLT for estimation of quantiles. The following lemma extends [Nadaraya, 1964] (Section 3) to the case of delta-sequences.

Lemma 5. *We assume that Conditions 1 and 2 hold. We have the CLT for the quantile estimators*

$$\sqrt{n}(\widehat{F}^{-1}(v) - F^{-1}(v)) \xrightarrow{\mathcal{D}} N_{F-1}, \tag{50}$$

$$\sqrt{n}(\widehat{G}^{-1}(v) - G^{-1}(v)) \xrightarrow{\mathcal{D}} N_{G^{-1}}, \tag{51}$$

as $n \to \infty$. Here, $N_{F^{-1}}(t)$ is a random variable with normal distribution characterized by its mean equal to zero and its variance equal to $\frac{v(1-v)}{f(F^{-1}(v))^2}$. Also, $N_{G^{-1}}(t)$ is a random variable with normal distribution characterized by its mean equal to zero and its variance equal to $\frac{\lambda v(1-v)}{g(G^{-1}(v))^2}$.

Proof of Lemma 5. By Condition 2 (e), we obtain that the cdf estimator (8) is continuously differentiable on the real positive set $\mathbb{R}^+$. Its derivative is equal to the pdf estimator (7). Thus, we can consider the Taylor expansion

$$\widehat{F}(\widehat{F}^{-1}(v)) - \widehat{F}(F^{-1}(v)) = (\widehat{F}^{-1}(v) - F^{-1}(v))\widehat{f}(\xi), \tag{52}$$

where ξ is a random variable between $F^{-1}(v)$ and $\widehat{F}^{-1}(v)$.

Let us first show that $\widehat{f}(\xi)$ tends to $f(F^{-1}(v))$ in probability. By the triangular inequality, we obtain

$$|\widehat{f}(\xi) - f(F^{-1}(v))| \leq |\widehat{f}(\xi) - f(\xi)| + |f(\xi) - f(F^{-1}(v))|. \tag{53}$$

By Lemma 4, we can deduce that ξ converges to $F^{-1}(v)$ in probability. If we introduce a sequence as $\epsilon > 0$ such that $\epsilon \to 0$, we also obtain

$$\mathbb{P}\big(|\widehat{f}(\xi) - f(\xi)| \leq \sup_{z \in [F^{-1}(v)-\epsilon, F^{-1}(v)+\epsilon]} |\widehat{f}(z) - f(z)|\big) \to 1. \tag{54}$$

By Lemma 3, we can deduce that

$$\sup_{z \in [F^{-1}(v)-\epsilon, F^{-1}(v)+\epsilon]} |\widehat{f}(z) - f(z)| \xrightarrow{\mathbb{P}} 0. \tag{55}$$

By Expressions (54) and (55), we can deduce that

$$|\widehat{f}(\xi) - f(\xi)| \xrightarrow{\mathbb{P}} 0. \tag{56}$$

Since ξ converges to $F^{-1}(v)$ in probability and f is continuous by Condition 2 (e), we obtain by an application of the continuous mapping theorem that

$$|f(\xi) - f(F^{-1}(v))| \xrightarrow{\mathbb{P}} 0. \tag{57}$$

By Expressions (53), (56) and (57), we can deduce that

$$|\widehat{f}(\xi) - f(F^{-1}(v))| \xrightarrow{\mathbb{P}} 0. \tag{58}$$

By Equation (52), we can deduce that a.s.

$$\sqrt{n}(\widehat{F}^{-1}(v) - F^{-1}(v)) = \sqrt{n}\frac{\widehat{F}(\widehat{F}^{-1}(v)) - \widehat{F}(F^{-1}(v))}{\widehat{f}(\xi)}. \tag{59}$$

By definition of the quantile estimator (9), we can deduce that

$$\sqrt{n}\frac{\widehat{F}(\widehat{F}^{-1}(v)) - \widehat{F}(F^{-1}(v))}{\widehat{f}(\xi)} = \sqrt{n}\frac{F(F^{-1}(v)) - \widehat{F}(F^{-1}(v))}{\widehat{f}(\xi)}. \tag{60}$$

By an application of Lemma 1 with Slutsky's theorem and Expression (58), we obtain

$$\sqrt{n}\frac{F(F^{-1}(v)) - \widehat{F}(F^{-1}(v))}{\widehat{f}(\xi)} \xrightarrow{\mathcal{D}} N_{F^{-1}}. \tag{61}$$

By Expressions (59), (60) and (61), we can deduce Expression (50). The same arguments with Condition 2 (b) yield Expression (51). □

We give now the proof of Theorem 1. This is based on a Taylor expansion, and an application of Lemmas 1 and 5.

Proof of Theorem 1. By the definitions of ROC curve (1) and its estimator (10), we can deduce

$$\widehat{ROC}(v) - ROC(v) = \widehat{G}(\widehat{F}^{-1}(1-v)) - G(F^{-1}(1-v)). \tag{62}$$

Then, we can deduce that

$$\widehat{G}(\widehat{F}^{-1}(1-v)) - G(F^{-1}(1-v)) = \widehat{G}(\widehat{F}^{-1}(1-v)) - \widehat{G}(F^{-1}(1-v)) \\ +\widehat{G}(F^{-1}(1-v)) - G(F^{-1}(1-v)). \tag{63}$$

By Condition 2 (e), we obtain that the cdf estimator (8) is continuously differentiable on the real positive set $\mathbb{R}^+$. Its derivative is equal to the pdf estimator (7). Thus, we can consider the Taylor expansion

$$\widehat{G}(\widehat{F}^{-1}(1-v)) - \widehat{G}(F^{-1}(1-v)) = (\widehat{F}^{-1}(1-v) - F^{-1}(1-v))\widehat{g}(\xi), \tag{64}$$

where ξ is a random variable between $F^{-1}(1-v)$ and $\widehat{F}^{-1}(1-v)$. By an application of Lemma 5 with Slutsky's theorem and Expression (58), we obtain

$$\sqrt{n}(\widehat{F}^{-1}(1-v) - F^{-1}(1-v))\widehat{g}(\xi) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{v(1-v)g(F^{-1}(1-v))^2}{f(F^{-1}(1-v))^2}\Big). \tag{65}$$

Here, $\mathcal{N}(m, \sigma^2)$ denotes a random variable which is normally distributed with mean m and variance σ^2. By Expressions (64) and (65), we can deduce that

$$\sqrt{n}(\widehat{G}(\widehat{F}^{-1}(1-v)) - \widehat{G}(F^{-1}(1-v))) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{v(1-v)g(F^{-1}(1-v))^2}{f(F^{-1}(1-v))^2}\Big). \tag{66}$$

By Lemma 1, we obtain

$$\sqrt{n}(\widehat{G}(F^{-1}(1-v)) - G(F^{-1}(1-v))) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \lambda G(F^{-1}(1-v))(1-G(F^{-1}(1-v))\Big). \tag{67}$$

By Condition 2 (c), we obtain

$$\mathrm{Cov}(\sqrt{n}(\widehat{G}(F^{-1}(1-v)) - G(F^{-1}(1-v))), \sqrt{n}(\widehat{G}(\widehat{F}^{-1}(1-v)) - \widehat{G}(F^{-1}(1-v)))) \to 0. \tag{68}$$

By Expressions (62), (63), (66), (67) and (68), we can deduce Expression (28). By an application of Lemmas 2, 3 and 4, with the definitions (12) and (14), we can deduce that $\widehat{V}_R(v)$ converges to $V_R(v)$ in probability. Then, we can deduce Expression (30) by Slutsky's theorem.

By the definitions of ODC (2) and its estimator (11), we can deduce

$$\widehat{ODC}(v) - ODC(v) = \widehat{F}(\widehat{G}^{-1}(v)) - F(G^{-1}(v)). \tag{69}$$

Then, we can deduce that

$$\widehat{F}(\widehat{G}^{-1}(v)) - F(G^{-1}(v)) = \widehat{F}(\widehat{G}^{-1}(v)) - \widehat{F}(G^{-1}(v)) + \widehat{F}(G^{-1}(v)) - F(G^{-1}(v)). \tag{70}$$

By Condition 2 (e), we obtain that the cdf estimator (8) is continuously differentiable on the real positive set $\mathbb{R}^+$. Its derivative is equal to the pdf estimator (7). Thus, we can consider the Taylor expansion

$$\widehat{F}(\widehat{G}^{-1}(v)) - \widehat{F}(G^{-1}(v)) = (\widehat{G}^{-1}(v) - G^{-1}(v))\widehat{f}(\xi), \tag{71}$$

where ξ is a random variable between $G^{-1}(v)$ and $\widehat{G}^{-1}(v)$. By an application of Lemma 5 with Slutsky's theorem and Expression (58), we obtain

$$\sqrt{n}(\widehat{G}^{-1}(v) - G^{-1}(v))\widehat{f}(\xi) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{\lambda v(1-v)f(G^{-1}(v))^2}{g(G^{-1}(v))^2}\Big). \tag{72}$$

By Expressions (71) and (72), we can deduce that

$$\sqrt{n}(\widehat{F}(\widehat{G}^{-1}(v)) - \widehat{F}(G^{-1}(v))) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{\lambda v(1-v)f(G^{-1}(v))^2}{g(G^{-1}(v))^2}\Big). \tag{73}$$

By Lemma 1, we obtain

$$\sqrt{n}(\widehat{F}(G^{-1}(v)) - F(G^{-1}(v))) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, F(G^{-1}(v))(1 - F(G^{-1}(v))\Big). \tag{74}$$

By Condition 2 (c), we obtain

$$\mathrm{Cov}(\sqrt{n}(\widehat{F}(G^{-1}(v)) - F(G^{-1}(v))), \sqrt{n}(\widehat{F}(\widehat{G}^{-1}(v)) - \widehat{F}(G^{-1}(v)))) \to 0. \tag{75}$$

By Expressions (69), (70), (73), (74) and (75), we can deduce Expression (29). By an application of Lemmas 2, 3 and 4, with the definitions (13) and (15), we can deduce that $\widehat{V}_O(v)$ converges to $V_O(v)$ in probability. Then, we can deduce Expression (31) by Slutsky's theorem. □

We give the CLT for multidimensional estimation of cdfs at different times in the following lemma. This extends Lemma 1, and its proof is based on the CLT normal convergence criterion (p. 307) from [Loeve, 1977] adapted to the multidimensional case.

Lemma 6. *We assume that Condition 1, Conditions 2 (a), (b) and (e) and Condition 3 (c) hold. We consider joint estimation at different times* (t_i) *for* $i = 1, \cdots, d$*, which satisfy* $0 < t_1 < t_2 < \ldots < t_d$ *and* $d \in \mathbb{N}$ *with* $d \geq 2$*. We have the CLT for the multidimensional cdf estimators*

$$\sqrt{n}((\widehat{F}(t_i))_{i=1}^d - (F(t_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_F(t_i))_{i=1}^d, \tag{76}$$

$$\sqrt{n}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_G(t_i))_{i=1}^d, \tag{77}$$

as $n \to \infty$*. Here,* $(N_F(t_i))_{i=1}^d$ *is a d-dimensional random vector with multidimensional normal distribution characterized by its variance* $\mathrm{Var}(N_F(t_i)) = F(t_i)(1-F(t_i))$ *and covariance* $\mathrm{Cov}(N_F(t_i), N_F(t_j)) = F(t_i)(1 - F(t_i))$ *for any* $i < j$*. Also,* $(N_G(t_i))_{i=1}^d$ *is a d-dimensional random vector with multidimensional normal distribution characterized by its variance* $\mathrm{Var}(N_G(t_i)) = \lambda G(t_i)(1-G(t_i))$ *and covariance* $\mathrm{Cov}(N_G(t_i), N_G(t_j)) = \lambda G(t_i)(1 - G(t_i))$ *for any* $i < j$*.*

Proof of Lemma 6. First, we can deduce by Condition 2 (e) that the cdfs at different times satisfy $0 < F(t_1) < \cdots < F(t_d) < 1$. We have that

$$\sqrt{n}((\widehat{F}(t_i))_{i=1}^d - F(t_i)_{i=1}^d) = \sqrt{n}((\widehat{F}(t_i))_{i=1}^d - (\mathbb{E}[\widehat{F}(t_i)])_{i=1}^d) + \sqrt{n}((\mathbb{E}[\widehat{F}(t_i)])_{i=1}^d - (F(t_i))_{i=1}^d).$$

By Condition 3 (c), we can deduce that the random vectors

$$\sqrt{n}((\widehat{F}(t_i))_{i=1}^d - (F(t_i))_{i=1}^d)$$

and

$$\sqrt{n}((\widehat{F}(t_i))_{i=1}^d - (\mathbb{E}[\widehat{F}(t_i)])_{i=1}^d)$$

converge to the same limit in distribution. By Conditions 1 and 2 (a), we can extend the arguments from the proof of Theorem 6 (iii) from [Watson and Leadbetter, 1964b]. Then, we obtain by the CLT normal convergence criterion (p. 307) from [Loeve, 1977] adapted to the multidimensional case that

$$\sqrt{n}((\widehat{F}(t_i))_{i=1}^d - (F(t_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_F(t_i))_{i=1}^d,$$

as $n \to \infty$. Here, $(N_F(t_i))_{i=1}^d$ is a random vector with a multidimensional normal distribution. By Lemma 1, we can deduce that $\mathrm{Var}(N_F(t_i)) = F(t_i)(1 - F(t_i))$. To determine $\mathrm{Cov}(N_F(t_i), N_F(t_j))$, we first obtain that

$$\mathrm{Cov}(\sqrt{n}(\widehat{F}(t_i) - \mathbb{E}[\widehat{F}(t_i)]), \sqrt{n}(\widehat{F}(t_j) - \mathbb{E}[\widehat{F}(t_j)])) \to \mathrm{Cov}(N_F(t_i), N_F(t_j)), \tag{78}$$

as $n \to \infty$. By Equation (8), we can deduce that

$$\begin{aligned} &\mathrm{Cov}(\sqrt{n}(\widehat{F}(t_i) - \mathbb{E}[\widehat{F}(t_i)]), \sqrt{n}(\widehat{F}(t_j) - \mathbb{E}[\widehat{F}(t_j)])) \\ = \ &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_0^{t_j} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_j} \widehat{f}(u)du\Big]\Big)\right). \end{aligned} \tag{79}$$

By linearity of the covariance and integral, we can deduce that

$$\begin{aligned} &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_0^{t_j} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_j} \widehat{f}(u)du\Big]\Big)\right) \\ = \ &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big)\right) \\ + \ &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_{t_i}^{t_j} \widehat{f}(u)du - \mathbb{E}\Big[\int_{t_i}^{t_j} \widehat{f}(u)du\Big]\Big)\right). \end{aligned} \tag{80}$$

By Lemma 1, we can deduce that

$$\begin{aligned} &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big)\right) \\ \to \ &F(t_i)(1 - F(t_i)), \end{aligned} \tag{81}$$

as $n \to \infty$. By definition of the pdf estimator (7), we can deduce that

$$\begin{aligned} &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big), \sqrt{n}\Big(\int_0^{t_i} \widehat{f}(u)du - \mathbb{E}\Big[\int_0^{t_i} \widehat{f}(u)du\Big]\Big)\right) \\ = \ &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du - \mathbb{E}\Big[\int_0^{t_i} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du\Big]\Big),\right. \\ &\left.\sqrt{n}\Big(\int_{t_i}^{t_j} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du - \mathbb{E}\Big[\int_{t_i}^{t_j} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du\Big]\Big)\right). \end{aligned} \tag{82}$$

By Condition 1 (b), we obtain

$$\begin{aligned} &\mathrm{Cov}\left(\sqrt{n}\Big(\int_0^{t_i} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du - \mathbb{E}\Big[\int_0^{t_i} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du\Big]\Big),\right. \\ &\left.\sqrt{n}\Big(\int_{t_i}^{t_j} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du - \mathbb{E}\Big[\int_{t_i}^{t_j} \frac{1}{n}\sum_{k=1}^n \delta(u - X_k)du\Big]\Big)\right) \to 0, \end{aligned} \tag{83}$$

as $n \to \infty$. Finally, we can deduce by Expressions (78), (79), (80), (81), (82) and (83) that

$$\mathrm{Cov}(\sqrt{n}(\widehat{F}(t_i) - \mathbb{E}[\widehat{F}(t_i)]), \sqrt{n}(\widehat{F}(t_j) - \mathbb{E}[\widehat{F}(t_j)])) \to F(t_i)(1 - F(t_i)),$$

as $n \to \infty$. Thus, we can deduce Expression (76).

To show Expression (77), we can first deduce by Condition 2 (e) that the cdfs at different times satisfy $0 < G(u_1) < \cdots < G(u_d) < 1$. We have that

$$\sqrt{n}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d) = \sqrt{m}\frac{\sqrt{n}}{\sqrt{m}}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d).$$

By Slutsky's theorem and Condition 2 (b), we can deduce that the random vectors

$$\sqrt{n}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d)$$

and

$$\sqrt{m}\sqrt{\lambda}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d)$$

converge to the same limit in distribution. Then, we have that

$$\begin{aligned} &\sqrt{m}\sqrt{\lambda}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d) \\ &= \sqrt{m}\sqrt{\lambda}((\widehat{G}(t_i))_{i=1}^d - \mathbb{E}[(\widehat{G}(t_i))_{i=1}^d]) + \sqrt{m}\sqrt{\lambda}(\mathbb{E}[(\widehat{G}(t_i))_{i=1}^d] - (G(t_i))_{i=1}^d). \end{aligned}$$

By Condition 3 (c), we can deduce that

$$\sqrt{m}\sqrt{\lambda}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d)$$

and

$$\sqrt{m}\sqrt{\lambda}((\widehat{G}(t_i))_{i=1}^d - \mathbb{E}[(\widehat{G}(t_i))_{i=1}^d])$$

have the same limit in distribution. By Conditions 1 and 2 (a), we can extend the arguments from the proof of Theorem 6 (iii) from [Watson and Leadbetter, 1964b]. Then, we obtain by the CLT normal convergence criterion (p. 307) from [Loeve, 1977] adapted to the multidimensional case that

$$\sqrt{n}((\widehat{G}(t_i))_{i=1}^d - (G(t_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_G(t_i))_{i=1}^d,$$

as $n \to \infty$. Here, $(N_G(t_i))_{i=1}^d$ is a random vector with a multidimensional normal distribution. By Lemma 1, we can deduce that $\mathrm{Var}(N_G(t_i)) = \lambda G(t_i)(1 - G(t_i))$. Finally, we can prove that $\mathrm{Cov}(N_G(t_i), N_G(t_j)) = \lambda G(t_i)(1 - G(t_i))$ with the arguments used in the proof of the F case. □

For a vector V, we denote its components as V^i. We give the CLT for multidimensional estimation of quantiles at different points in the following lemma. This extends Lemma 5, and its proof is based on Lemma 6.

Lemma 7. *We assume that Conditions 1, 2 and Condition 3 (c) hold. We consider joint estimation at different points (u_i) for $i = 1, \cdots, d$, which satisfy $0 < u_1 < u_2 < \ldots < u_d < 1$ and $d \in \mathbb{N}$ with $d \geq 2$. We have the multidimensional CLT for the quantile estimators*

$$\sqrt{n}((\widehat{F}^{-1}(u_i))_{i=1}^d - (F^{-1}(u_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_{F^{-1}}(u_i))_{i=1}^d, \tag{84}$$

$$\sqrt{n}((\widehat{G}^{-1}(u_i))_{i=1}^d - (G^{-1}(u_i))_{i=1}^d) \xrightarrow{\mathcal{D}} (N_{G^{-1}}(u_i))_{i=1}^d, \tag{85}$$

as $n \to \infty$. Here, $(N_{F^{-1}}(u_i))_{i=1}^d$ is a d-dimensional random vector with multidimensional normal distribution characterized by its variance $\mathrm{Var}(N_{F^{-1}}(u_i)) = \frac{u_i(1-u_i)}{f(F^{-1}(u_i))^2}$ and covariance

$$\mathrm{Cov}(N_{F^{-1}}(u_i), N_{F^{-1}}(u_j)) = \frac{u_i(1 - u_i)}{f(F^{-1}(u_i))^2},$$

for any $i < j$. Also, $(N_{G^{-1}}(u_i))_{i=1}^d$ is a d-dimensional random vector with multidimensional normal distribution characterized by its variance $\mathrm{Var}(N_{G^{-1}}(u_i)) = \lambda\frac{u_i(1-u_i)}{g(G^{-1}(u_i))^2}$ and covariance

$$\mathrm{Cov}(N_{G^{-1}}(u_i), N_{G^{-1}}(u_j)) = \lambda\frac{u_i(1 - u_i)}{g(G^{-1}(u_i))^2},$$

for any $i < j$.

Proof of Lemma 7. We can consider the multidimensional Taylor expansion

$$(\widehat{F}(\widehat{F}^{-1}(u_i)))_{i=1}^d - (\widehat{F}(F^{-1}(u_i)))_{i=1}^d = ((\widehat{F}^{-1}(u_i))_{i=1}^d - (F^{-1}(u_i))_{i=1}^d)\widehat{f}(\xi_i), \tag{86}$$

where ξ_i are random variables between $F^{-1}(u_i)$ and $\widehat{F}^{-1}(u_i)$.

Let us first show that $(\widehat{f}(\xi_i))_{i=1}^d$ tends to $(f(F^{-1}(u_i)))_{i=1}^d$ in probability. By the triangular inequality and the definition of norm, we obtain

$$||(\widehat{f}(\xi_i))_{i=1}^d - (f(F^{-1}(u_i)))_{i=1}^d|| \leq \sum_{i=1}^d |\widehat{f}(\xi_i) - f(\xi_i)| + |f(\xi_i) - f(F^{-1}(u_i))|. \tag{87}$$

By Lemma 4, we can deduce that ξ_i converges to $F^{-1}(u_i)$ in probability for $i = 1, \cdots, d$. If we introduce for $i = 1, \cdots, d$ a sequence as $\epsilon_i > 0$ such that $\epsilon_i \to 0$, we also obtain

$$\mathbb{P}\big(|\widehat{f}(\xi_i) - f(\xi_i)| \leq \sup_{z \in [F^{-1}(u_i)-\epsilon_i, F^{-1}(u_i)+\epsilon_i]} |\widehat{f}(z) - f(z)|\big) \to 1. \tag{88}$$

By Lemma 3, we can deduce that

$$\sup_{z \in [F^{-1}(u_i)-\epsilon_i, F^{-1}(u_i)+\epsilon_i]} |\widehat{f}(z) - f(z)| \xrightarrow{\mathbb{P}} 0. \tag{89}$$

By Expressions (88) and (89), we can deduce that

$$|\widehat{f}(\xi_i) - f(\xi_i)| \xrightarrow{\mathbb{P}} 0. \tag{90}$$

Since ξ_i converges to $F^{-1}(u_i)$ in probability and f is continuous by Condition 2 (e), we obtain by an application of the continuous mapping theorem that

$$|f(\xi_i) - f(F^{-1}(u_i))| \xrightarrow{\mathbb{P}} 0 \tag{91}$$

By Expressions (87), (90) and (91), we can deduce that

$$||(\widehat{f}(\xi_i))_{i=1}^d - (f(F^{-1}(u_i)))_{i=1}^d|| \xrightarrow{\mathbb{P}} 0. \tag{92}$$

By Equation (86), we can deduce that a.s.

$$\sqrt{n}((\widehat{F}^{-1}(u_i))_{i=1}^d - (F^{-1}(u_i))_{i=1}^d) = \sqrt{n}\frac{(\widehat{F}(\widehat{F}^{-1}(u_i)))_{i=1}^d - (\widehat{F}(F^{-1}(u_i)))_{i=1}^d}{(\widehat{f}(\xi_i))_{i=1}^d}. \tag{93}$$

Here, the multidimensional division operation is componentwise, i.e. $\frac{U}{V} = \big(\frac{U^i}{V^i}\big)_{i=1}^d$. By definition of the quantile estimator (9), we can deduce that

$$\sqrt{n}\frac{(\widehat{F}(\widehat{F}^{-1}(u_i)))_{i=1}^d - (\widehat{F}(F^{-1}(u_i)))_{i=1}^d}{(\widehat{f}(\xi_i))_{i=1}^d} = \sqrt{n}\frac{(F(F^{-1}(u_i)))_{i=1}^d - (\widehat{F}(F^{-1}(u_i)))_{i=1}^d}{(\widehat{f}(\xi_i))_{i=1}^d}. \tag{94}$$

By an application of Lemma 6 with Slutsky's theorem and Expression (92), we obtain

$$\sqrt{n}\frac{(F(F^{-1}(u_i)))_{i=1}^d - (\widehat{F}(F^{-1}(u_i)))_{i=1}^d}{(\widehat{f}(\xi_i))_{i=1}^d} \xrightarrow{\mathcal{D}} (N_{F^{-1}}(u_i))_{i=1}^d. \tag{95}$$

By Expressions (93), (94) and (95), we can deduce Expression (84). The same arguments with Condition 2 (b) yield Expression (85).

□

In the following lemma, we give the CLT for the process of ROC curve statistics, which converges to the limit Gaussian process $(N_R(u))_{\{0<u<v\}}$. We also consider the ODC case. This extends the proof of Theorem 1.

Lemma 8. *We assume that Conditions 1, 2 and Condition 3 (c) hold. We also assume that* $0 < v_0 < v < 1$. *We have the CLT for the process of ROC curve statistics which converges to the the limit Gaussian process, i.e.*

$$\sqrt{n}\big(\widehat{ROC}(u) - ROC(u)\big)_{\{v_0<u<v\}} \xrightarrow{\mathcal{D}} (N_R(u))_{\{v_0<u<v\}}, \tag{96}$$

$$\sqrt{n}\big(\widehat{ODC}(u) - ODC(u)\big)_{\{v_0<u<v\}} \xrightarrow{\mathcal{D}} (N_O(u))_{\{v_0<u<v\}}, \tag{97}$$

as $n \to \infty$. *Here, the Gaussian process* $(N_R(u))_{\{v_0<u<v\}}$ *is characterized by its variance* $\mathrm{Var}(N_R(u)) = V_R(u)$, *and its covariance*

$$\mathrm{Cov}(N_R(u), N_R(w)) = V_R(u),$$

for $v_0 < u < w < v$. *Also, the Gaussian process* $(N_O(u))_{\{v_0<u<v\}}$ *is characterized by its variance* $\mathrm{Var}(N_O(u)) = V_O(u)$, *and its covariance*

$$\mathrm{Cov}(N_O(u), N_O(w)) = V_O(u),$$

for $v_0 < u < w < v$.

Proof of Lemma 8. To show that $(N_R(u))_{\{0<u<v\}}$ is a Gaussian process limit, it is sufficient to prove that

$$\sqrt{n}\big(\widehat{ROC}(u_i) - ROC(u_i)\big)_{i=1}^d \xrightarrow{\mathcal{D}} (N_R(u_i))_{i=1}^d, \tag{98}$$

as $n \to \infty$ and where $v_0 < u_1 < u_2 < \ldots < u_d < v$.

By the definitions of ROC curve (1) and its estimator (10), we can deduce

$$\sqrt{n}\big(\widehat{ROC}(u_i) - ROC(u_i)\big)_{i=1}^{d} = \sqrt{n}\big(\widehat{G}(\widehat{F}^{-1}(1-u_i)) - G(F^{-1}(1-u_i))\big)_{i=1}^{d}. \quad (99)$$

Then, we can deduce that

$$\begin{aligned} &\sqrt{n}\big(\widehat{G}(\widehat{F}^{-1}(1-u_i)) - G(F^{-1}(1-u_i))\big)_{i=1}^{d} \quad (100)\\ =&\sqrt{n}\Big(\widehat{G}(\widehat{F}^{-1}(1-u_i)) - \widehat{G}(F^{-1}(1-u_i))\big)_{i=1}^{d}\\ &+\sqrt{n}\big(\widehat{G}(F^{-1}(1-u_i)) - G(F^{-1}(1-u_i))\big)_{i=1}^{d}. \end{aligned}$$

Then, we can consider the multidimensional Taylor expansion

$$\sqrt{n}\big(\widehat{G}(\widehat{F}^{-1}(1-u_i)) - \widehat{G}(F^{-1}(1-u_i))\big)_{i=1}^{d} = \sqrt{n}\big((\widehat{F}^{-1}(1-u_i) - F^{-1}(1-u_i))\widehat{g}(\xi_i)\big)_{i=1}^{d}. \quad (101)$$

Here, ξ_i is a random variable between $F^{-1}(1-u_i)$ and $\widehat{F}^{-1}(1-u_i)$ for $i = 1, \cdots, d$. By an application of Lemma 7 with Slutsky's theorem and Expression (58), we obtain

$$\sqrt{n}\big((\widehat{F}^{-1}(1-u_i) - F^{-1}(1-u_i))\widehat{g}(\xi_i)\big)_{i=1}^{d} \xrightarrow{\mathcal{D}} (g(F^{-1}(1-u_i))N_{F^{-1}}(1-u_i))_{i=1}^{d}. \quad (102)$$

By Expressions (101) and (102), we can deduce that

$$\sqrt{n}\big(\widehat{G}(\widehat{F}^{-1}(1-u_i)) - \widehat{G}(F^{-1}(1-u_i))\big)_{i=1}^{d} \xrightarrow{\mathcal{D}} (g(F^{-1}(1-u_i))N_{F^{-1}}(1-u_i))_{i=1}^{d}. \quad (103)$$

By Lemma 6, we obtain

$$\sqrt{n}\big(\widehat{G}(F^{-1}(1-u_i)) - G(F^{-1}(1-u_i))\big)_{i=1}^{d} \xrightarrow{\mathcal{D}} (N_G(F^{-1}(1-u_i)))_{i=1}^{d}. \quad (104)$$

By Condition 2 (c), we obtain

$$\mathrm{Cov}(\sqrt{n}(\widehat{G}(F^{-1}(1-u_i)) - G(F^{-1}(1-u_i))), \sqrt{n}(\widehat{G}(\widehat{F}^{-1}(1-u_j)) - \widehat{G}(F^{-1}(1-u_j)))) \to 0, (105)$$

for $i = 1, \cdots, d$ and $j = 1, \cdots, d$. By Expressions (99), (100), (103), (104) and (105), we can deduce Expression (96). The same arguments yield Expression (97). $\square$

In what follows, we give the CLT of the integral of the ROC curve process. We also consider the ODC case. The proof is based on the Gaussian process property of the limit obtained in Lemma 8.

Lemma 9. *We assume that Conditions 1, 2 and Condition 3 (c) hold. We also assume that* $0 < v_0 < v < 1$*. We have the CLT for the integral*

$$\int_{v_0}^{v} \sqrt{n}(\widehat{ROC}(u) - ROC(u))du \xrightarrow{\mathcal{D}} \int_{v_0}^{v} N_R(u)du, \tag{106}$$

$$\int_{v_0}^{v} \sqrt{n}(\widehat{ODC}(u) - ODC(u))du \xrightarrow{\mathcal{D}} \int_{v_0}^{v} N_O(u)du, \tag{107}$$

as $n \to \infty$*. Here, the first limit is a random variable with normal distribution which is characterized by its variance*

$$\mathrm{Var}\Big(\int_{v_0}^{v} N_R(u)du\Big) = \int_{v_0}^{v} (1 + 2(v-u))V_R(u)du. \tag{108}$$

Also, the second limit is a random variable with normal distribution which is characterized by its variance

$$\mathrm{Var}\Big(\int_{v_0}^{v} N_O(u)du\Big) = \int_{v_0}^{v} (1 + 2(v-u))V_O(u)du.$$

Proof of Lemma 9. We obtain Expression (106) and the fact that the limit is a random variable with normal distribution with the Gaussian process property of the limit obtained in Lemma 8. By variance and covariance properties of a Gaussian process, we obtain that

$$\mathrm{Var}\Big(\int_{v_0}^{v} N_R(u)du\Big) = \int_{v_0}^{v} \mathrm{Var}(N_R(u))du + \int_{v_0}^{v}\int_{v_0}^{v} \mathrm{Cov}(N_R(u), N_R(w))dudw. \tag{109}$$

From Lemma 8, we can deduce that

$$\int_{v_0}^{v} \mathrm{Var}(N_R(u))du = \int_{v_0}^{v} V_R(u)du. \tag{110}$$

From Lemma 8 and Tonelli's theorem, we obtain that

$$\int_{v_0}^{v}\int_{v_0}^{v} \mathrm{Cov}(N_R(u), N_R(w))dudw = 2\int_{v_0}^{v} (v-u)V_R(u)du. \tag{111}$$

We can deduce Equation (108) with Equations (109), (110) and (111). The same arguments yield Expression (107). □

We give now the uniform consistency of the quantile estimators. The following lemma extends Lemma 4.

Lemma 10. *We assume that Conditions 1, 2 (b) and 3 (b) hold. We have the uniform consistency of the quantile estimators*

$$\sup_{u\in[1-v,1-v_0]} \left|\widehat{F}^{-1}(u) - F^{-1}(u)\right| \xrightarrow{\mathbb{P}} 0 \text{ and } \sup_{u\in[v_0,v]} \left|\widehat{G}^{-1}(u) - G^{-1}(u)\right| \xrightarrow{\mathbb{P}} 0, \tag{112}$$

as $n \to \infty$.

Proof of Lemma 10. For $\epsilon > 0$ and $u \in [1-v, 1-v_0]$, we consider

$$\delta(u) = \min(F(F^{-1}(u)+\epsilon) - u, u - F(F^{-1}(u)-\epsilon)),$$

$$\delta = \inf_{u\in[1-v,1-v_0]} \delta(u)$$

By Condition 3 (b), we can deduce that $\delta > 0$. By definition of δ, we obtain

$$\mathbb{P}\Big(\sup_{u\in[1-v,1-v_0]} \left|\widehat{F}^{-1}(u) - F^{-1}(u)\right| > \epsilon\Big) \le \mathbb{P}\Big(\sup_{u\in[1-v,1-v_0]} \left|F(\widehat{F}^{-1}(u)) - u\right| > \delta\Big). \tag{113}$$

By definition of the quantile estimator (9), we obtain

$$\sup_{u\in[1-v,1-v_0]} \left|F(\widehat{F}^{-1}(u)) - u\right| = \sup_{u\in[1-v,1-v_0]} \left|F(\widehat{F}^{-1}(u)) - \widehat{F}(\widehat{F}^{-1}(u))\right|. \tag{114}$$

By bounding the term in the right hand-side of Equation (114), we obtain

$$\sup_{u\in[1-v,1-v_0]} \left|F(\widehat{F}^{-1}(u)) - \widehat{F}(\widehat{F}^{-1}(u))\right| \le \sup_{t\in\mathbb{R}^+} \left|F(t) - \widehat{F}(t)\right|. \tag{115}$$

By Lemma 2 with Expressions (113), (114) and (115), we can deduce that

$$\sup_{u\in[1-v,1-v_0]} \left|\widehat{F}^{-1}(u) - F^{-1}(u)\right| \xrightarrow{\mathbb{P}} 0.$$

The same arguments with Condition 2 (b) yield

$$\sup_{u\in[v_0,v]} \left|\widehat{G}^{-1}(u) - G^{-1}(u)\right| \xrightarrow{\mathbb{P}} 0.$$

□

We give now the uniform consistency of the ROC curve and ODC estimators. The following lemma is a direct application of Lemmas 2 and 10.

Lemma 11. *We assume that Conditions 1, 2 (b) and 3 (b) hold. We have the uniform consistency of the ROC curve and ODC curve estimators*

$$\sup_{u\in[1-v,1-v_0]} \left|\widehat{ROC}(u) - ROC(u)\right| \xrightarrow{\mathbb{P}} 0 \text{ and } \sup_{u\in[v_0,v]} \left|\widehat{ODC}(u) - ODC(u)\right| \xrightarrow{\mathbb{P}} 0, \tag{116}$$

as $n \to \infty$.

Proof of Lemma 11. This is an application of Lemmas 2 and 10, with Slutsky's theorem. □

In what follows, we give the proof of Theorem 2, which is mainly based on an application of Lemma 9.

Proof of Theorem 2. By the definition of the partial area under the ROC curve (3) and its estimator (16), we can deduce

$$\sqrt{n}\Big(\widehat{pROC}(v_0, v) - pROC(v_0, v)\Big) = \sqrt{n}\Big(\sum_{k=0}^{M-1} \widehat{ROC}(v_k)\Delta - \int_{v_0}^{v} ROC(u)du\Big). \tag{117}$$

Then, the right-hand side in Equation (117) can be rewritten as

$$\sqrt{n}\Big(\sum_{k=0}^{M-1} \widehat{ROC}(v_k)\Delta - \int_{v_0}^{v} ROC(u)du\Big) = \sqrt{n}\Big(\sum_{k=0}^{M-1} \widehat{ROC}(v_k)\Delta - \int_{v_0}^{v} \widehat{ROC}(u)du\Big) + \int_{v_0}^{v} \sqrt{n}(\widehat{ROC}(u) - ROC(u))du. \tag{118}$$

First, we can deduce by Lemma 9 that

$$\int_{v_0}^{v} \sqrt{n}(\widehat{ROC}(u) - ROC(u))du \xrightarrow{\mathcal{D}} \int_{v_0}^{v} N_R(u)du. \tag{119}$$

By an application of Lemma 11, we obtain

$$\sqrt{n}\Big(\sum_{k=0}^{M-1} \widehat{ROC}(v_k)\Delta - \int_{v_0}^{v} \widehat{ROC}(u)du\Big) - \sqrt{n}\Big(\sum_{k=0}^{M-1} ROC(v_k)\Delta - \int_{v_0}^{v} ROC(u)du\Big) \xrightarrow{\mathbb{P}} 0. \tag{120}$$

By the triangular inequality, we can deduce

$$\sqrt{n}\Big|\sum_{k=0}^{M-1} ROC(v_k)\Delta - \int_{v_0}^{v} ROC(u)du\Big| \le \sqrt{n}\sum_{k=0}^{M-1} \int_{v_k}^{v_{k+1}} \Big|ROC(v_k) - ROC(u)\Big|du. \tag{121}$$

We obtain by Conditions 2 (e) and 3 (a) that

$$\sqrt{n}\sum_{k=0}^{M-1}\int_{v_k}^{v_{k+1}}\Big|ROC(v_k)-ROC(u)\Big|du \xrightarrow{\mathbb{P}} 0. \tag{122}$$

We can deduce from Expressions (120), (121) and (122) that

$$\sqrt{n}\Big(\sum_{k=0}^{M-1}\widehat{ROC}(v_k)\Delta-\int_{v_0}^{v}\widehat{ROC}(u)du\Big) \xrightarrow{\mathbb{P}} 0. \tag{123}$$

Finally, we can deduce Expression (32) by Expressions (117), (118), (119) and (123). By similar arguments, we can show that $\widehat{V}_{pO}(v_0,v)$ converges to $V_{pO}(v_0,v)$ in probability. Then, we can deduce Expression (33) by Slutsky's theorem. The same arguments yield Expressions (34) and (35). □

In what follows, we give the proof of Theorem 3, which is based on a direct application of Theorem 2.

Proof of Theorem 3. The theorem is proven with the same arguments from the proof of Theorem 2. □