Nonparametric estimation of hitting-time variance

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Abstract: We consider estimation of hitting-time variance, i.e. the explicit solution in the inverse first-hitting time problem of a continuous martingale to a constant boundary. The nonparametric estimation is based on delta-sequences. We also consider tuning parameter estimation related to the boundary. We characterize feasible statistics induced by central limit theory for the estimation procedure. A numerical simulation corroborates the asymptotic theory. An empirical application to financial data documents that the volatility is periodic at the duration scale. This can be explained by the endogeneity of transaction times.

Keywords and phrases: statistics for stochastic processes, hitting time, inverse problem, nonparametric estimation, volatility periodicity.

1. Introduction

This paper focuses on first hitting time (FHT), i.e. the time when a stochastic process crosses a boundary. More specifically, this paper concerns estimation of hitting-time variance, i.e. the explicit solution in the inverse first-hitting time (IFHT) problem of a continuous martingale to a constant boundary. The IFHT problem determines the variance function such that the FHT of a standard Brownian motion, which is time changed by this variance integral, to the boundary has a given distribution. It is a new problem, which was introduced in the more probabilistic paper Potiron (2023). The probabilistic paper shows that the hitting-time variance is equal to the ratio of probability distribution function (pdf) over a function of cumulative distribution function (cdf), which depends on the boundary value. The novelty in this more statistics paper is that we consider estimation of hitting-time variance. The nonparametric estimation is based on delta-sequences. As far as the authors know, estimation of time-varying variance of a continuous local martingale is completely novel to the literature on FHT in statistics for stochastic processes.

In the literature of statistics for stochastic processes, the FHT of a continuous local martingale have applications when estimating the quadratic variation of a continuous local martingale based on endogenous observations. This is the so-called integrated volatility problem. In these models, endogenous observations are often generated by the FHT of a local martingale to a boundary process.

Fukasawa (2010) considers the FHT to a symmetric two-sided boundary. Robert and Rosenbaum (2011) and Robert and Rosenbaum (2012) introduce the model with uncertainty zones in which the two-sided boundary is dynamic. Fukasawa and Rosenbaum (2012) consider the FHT to a two-sided boundary, which is non-symmetric. Li et al. (2014) consider a more general framework of endogeneity allowing for FHT. Renault, Van der Heijden and Werker (2014) consider the mixed FHT of the sum of a Brownian motion and a positive linear drift. Potiron and Mykland (2017) estimate the quadratic covariation between two local martingales. Potiron and Mykland (2020) (see Section 4.4) consider an extension of the model with uncertinty zones. Cui (2024) proposes inference on estimation of quadratic variation based on FHT.

Although the IFHT problem is new, it has useful applications to model financial data. We can reinterpret the hitting-time variance as the squared volatility of an efficient price. Figure 6 and Table 9 show the estimated squared volatility. We find that the squared volatility is periodic and follows an inverse J-shape pattern. Such behavior of the squared volatility is empirically well documented at the scale of a day. However, our results are at the scale of a duration, which on average corresponds to 0.15 seconds. This can be explained by the endogeneity of the transaction times.

The application of the FHT in statistics can be traced back to the Kolmogorov-Smirnov statistic. The primary application of the FHT can be found in sequential analysis. At first, the focus was on the FHT of a random walk. Due to the complexity of the problem, the literature often relies on the FHT of a Brownian motion (see Gut (1974), Woodroofe (1976), Woodroofe (1977)), Lai and Siegmund (1977)), Lai and Siegmund (1979)) and Siegmund (1986)). In survival analysis, Matthews, Farewell and Pyke (1985) show that tests for constant hazard involve the FHT of an Ornstein-Uhlenbeck process. Butler and Huzurbazar (1997) consider a Bayesian approach for the FHT of a semi-Markovian process. Eaton and Whitmore (1977) study the application of the FHT for hospital stay. Aalen and Gjessing (2001) consider the FHT of a Markovian process. Detailed reviews on the FHT can be found in Lee and Whitmore (2006) and Lawless (2011) (Section 11.5, pp. 518-523).

There are also applications of the FHT in econometrics. They can be traced back to Lancaster (1979) mixed proportional hazards model. More recently, Abbring (2012) studies the mixed FHT of a Levy process. Liu (2020) considers the FHT of a dependent Levy process. Botosaru (2020) develops nonparametric estimation for the FHT. Abbring and Salimans (2021) computes the likelihood of a mixed FHT. Lin and Liu (2021) propose a dependent FHT. Kim (2023) develops a tractable approach based on bounds for the FHT. Alvarez, Borovičková and Shimer (2024) develop an economic model of transitions in and out of employment with the FHT. There are also applications of FHT in mathematical finance. Roberts and Shortland (1997) and Borovkov and Novikov (2002) provide an application of the FHT for the pricing of barrier options.

Our aim in this paper is to estimate nonparametrically the hitting-time variance. Since the formula of the hitting-time variance is similar to hazard functions, we will rely on estimation procedures from survival analysis. We consider nonparametric estimation of distribution (Equations (2.1)-(2.2), p. 176) from Watson and Leadbetter (1964a), based on delta-sequences. These delta-sequences are a sequence of functions approaching the Dirac function asymptotically which covers several types of smoothing estimation including the kernel estimation. We also give estimation of functions of hitting-time variance by plugging the change hazard function estimator into the function. We also propose estimation of cumulative functions of hitting-time variance based on empirical cumulative functions of hitting-time variance estimates. These results are obtained primarily to deduce an estimator of boundary, which is a tuning parameter. There are also of interest in their own right. Finally, we consider estimation of boundary, which normalizes the estimator for cumulative functions of hitting-time variance. We characterize feasible statistics induced by central limit theory (CLT) for hitting-time variance estimation, function of hitting-time variance estimation and cumulative function of hitting-time variance estimation. We also obtain consistency in the estimation of boundary tuning parameter, and that asymptotic properties of the feasible statistics are preserved when the boundary tuning parameter is estimated.

The estimation of standard hazard functions, i.e. the ratio of pdf over unity minus cdf, is close to nonparametric estimation of density, and there are many methods available. Watson and Leadbetter (1964b) provide delta-sequence based estimation of distribution and a first standard hazard function estimator (1.1) (p. 102), which is equal to the ratio of pdf estimates over unity minus cdf estimates. The respective CLT of these three estimators is available in their Theorem 5 (p. 111), Theorem 6 (iii) (p. 112) and Theorem 7 (p. 114). They consider another estimator of standard hazard function based on convolution (below Equation (1.1), p. 102). The ratio estimator has no asymptotic bias and the convolution estimator has an asymptotic bias, whereas both estimators have the same asymptotic variance (see Rice and Rosenblatt (1976), Watson and Leadbetter (1964b) (Theorem 7), Müller and Wang (1990) (Lemma 1)). Moreover, the ratio estimator is more flexible as it does not require to be in the class of cdfs considered in Equation (2.2) (p. 103) from Watson and Leadbetter (1964b). Eventually, the convolution estimator has prevailed owing to its theoretical tractability in finite sample, i.e. exact mean square errors (MSE) available, and its aesthetic superiority over the ratio estimator. Yet, the cumulative hitting-time variance is no longer equal to minus log of unity minus cdf as in Equation (1.3) from Watson and Leadbetter (1964a) (p. 175) and thus it is not possible to adapt the convolution estimator. Accordingly, we choose to adapt the ratio estimator to our hitting-time variance case.

Our theoretical contribution can be reexpressed as the estimation of standard hazard function, in case when the denominator is replaced by a known function of the cdf. More specifically, our CLT (see Section 4) extends Theorem 7 (p. 114) from Watson and Leadbetter (1964b) in four directions. (a) They consider standard hazard function estimation, but we consider hitting-time variance estimation. (b) We develop estimation of functions of hitting-time variance. (c) We study estimation for cumulative functions of hitting-time variance. (d) We give consistent estimators of the asymptotic variance and feasible CLT, none of which are provided in Theorem 7. We obtain the square root of the number of observations over the square root of time-averaged squared delta-sequence as rate of convergence for (a), (b) and (c), which is also equal to the rate of convergence in Theorem 7. However, the asymptotic variance is altered as it is equal to a more complicated form in (a) and (b), and a cumulative asymptotic variance of (b) in (c). Our proofs also extend their machinery as a more complicated function form in the denominator of the hitting-time variance implies deeper calculation in (a), the use of a Taylor expansion in (b), and uniform arguments in (c).

We discuss about an appropriate choice of the delta-sequence and the tuning parameters (see Section 5). We recommend to use the triangular kernel as it is the nonnegative kernel with the smallest MSE in Singpurwalla and Wong (1983) (see Table 1) in the case of exponential cdf. Müller and Wang (1990) give a general principle to obtain local bandwidth, i.e. minimizing the local MSE in case of the hazard estimator based on ordered sample values. Lo, Mack and Wang (1989) give the MSE in the case of the hazard ratio estimator, but they do not minimize it with respect to the bandwidth. We have that the asymptotic variance diverges at the extremities, and the support of the delta-sequence can exceed the available range of data. Thus, there may be bias problems at the extremities. The same problem appears when estimating the hazard function and is referred as extremity effects. Müller and Wang (1994) propose hazard function estimation with varying kernels and bandwidths in their Equation (2.2) (p. 62), based on Gasser and Müller (1979) earlier proposal. In their algorithm, they recommend to start with $b = \frac{1}{8n^{1/5}}$, which can be expressed in terms of our tuning parameter as $\alpha_n = 8n^{1/5}$. A rigorous study for the MSE of hitting-time variance, the choice of local efficient bandwidth and delta-sequence is above the scope of this paper and left for future work. However, we will find in our numerical simulation (see Section 6) that a larger tuning parameter value, i.e. $\alpha_n = 8n^{\frac{9}{10}}$, will give better results for all the cases we have considered. The numerical simulation also corroborates the asymptotic theory when all the tuning parameters are feasible.

Finally, we conduct an empirical application to financial data (see Section 7). The dataset contains information about transactions of a stock price and the transaction times. More specifically, we restrict to the transactions that correspond to a price change. By duration, we mean the time increment between two consecutive transactions in the dataset. We consider that the next transaction occurs whenever the absolute value of the efficient price increment starting at the previous transaction time hits a constant boundary. The model is based on the FHT. We rely on durations of the stock Apple Inc. (AAPL), which is traded daily from 9:30 AM to 4:00 PM on the Nasdaq stock exchange during the year 2017.

Since our empirical application is based on financial data, we do observe the data fully. However, the case of censored data may be of interest for applications in survival analysis and medicine. Extensions of the ratio hazard estimator to censored data are available in Blum and Susarla (1980), Földes, Rejto and

Winter (1981) and Lo, Mack and Wang (1989). They are based on Kaplan-Meier estimation (see Kaplan and Meier (1958)). Thus, we recommend to use the Kaplan-Meier estimator, rather than the empirical estimator. Then, we can use delta-sequences based on the Kaplan-Meier estimator. We conjecture that the theory holds. However, investigating this adaptation is above the scope of this paper and left for future work.

The remainder of this paper is organized as follows. The set-up for the IFHT problem is exposed in Section 2. In Sections 3 and 4, we provide estimation and theory. A discussion including implementation is given in Section 5. A numerical simulation can be found in Section 6. An empirical application is conducted in Section 7. We conclude in Section 8. The supplementary materials include the following. We specify the hitting-time variance for some standard parametric pdfs in Section 9. Finally, all proofs of the theory can be found in Section 10.

2. Set-up: IFHT problem

In this section, we introduce the IFHT problem. For more details on the IFHT problem, see Potiron (2023) (see Section 3). More specifically, we consider the particular case when the quadratic variation of the process is absolutely continuous, nonrandom, and the boundary is one-sided (see Section 3.1.1).

We consider the complete stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F}, \mathbf{F})$, where $\mathcal{F}$ is a σ-field and $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is a filtration. We consider a continuous stochastic process Z, which starts from 0 and is $\mathbf{F}$-adapted. We define $\mathbb{R}_*^+$ as the positive real space, without 0. We consider a boundary $g \in \mathbb{R}_*^+$, which is constant and positive. We define the FHT of the process Z to the boundary g as

$$\mathrm{T}_g^Z \quad = \quad \inf\{t \geq 0 \text{ s.t. } Z_t \geq g\}. \tag{1}$$

We define the cdf of the FHT T_g^Z as P_g^Z, which satisfies

$$P_g^Z(t) \quad = \quad \mathbb{P}(\mathrm{T}_g^Z \leq t) \text{ for any } t \geq 0. \tag{2}$$

When the cdf is absolutely continuous, we can also define its pdf as f_g^Z, which satisfies $f_g^Z(t) = \frac{dP_g^Z(t)}{dt}$ a.e..

We introduce the IFHT problem in what follows. The IFHT problem determines the variance function such that the FHT of a standard Brownian motion, which is time changed by this variance integral, to the boundary has a given distribution. More specifically, we consider the standard Brownian motion W, which is $\mathbf{F}$-adapted. We also consider the variance function $\sigma^2 : \mathbb{R}^+ \to \mathbb{R}_*^+$. For a given cdf F, we determine the variance σ^2, of the local martingale X defined as

$$X_t = \int_0^t \sigma_s dW_s \text{ for any } t \geq 0, \tag{3}$$

such that $P_g^X(t) = F(t)$ for any $t \geq 0$.

Potiron (2023) shows that the cdf F is absolutely continuous. Thus, we can define its pdf as f, which satisfies $f(t) = \frac{dF(t)}{dt}$ a.e.. Potiron (2023) (see Theorem 5) shows that the hitting-time variance is equal to the ratio of pdf over a function of cdf, i.e.

$$\sigma_{t,g}^2 = \frac{f(t)}{f_g^Z((P_g^Z)^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{for any } t \geq 0. \tag{4}$$

Here, f_g^Z and P_g^Z are Levy pdf and cdf with scale parameter equal to the square of the boundary g^2.

Our aim in this paper is to estimate the hitting-time variance (4) nonparametrically. Since the formula of the hitting-time variance is similar to hazard functions, we will rely on estimation procedures from survival analysis.

3. Estimation

In this section, we introduce a two-step procedure to estimate nonparametrically the hitting-time variance when the pdf f and the cdf F are unknown. We also introduce nonparametric estimation for functions, time-averaged functions of hitting-time variance, and boundary.

We prefer most of the time not to write explicitly the dependence on n. We consider estimation based on n observations of FHT. More specifically, for any $i = 1, \cdots, n$ we consider standard Brownian motions $W^{(i)}$, which are independent and $\mathbf{F}$-adapted. We define for any $i = 1, \cdots, n$ and any $t \geq 0$ the processes $X_t^{(i)} = \int_0^t \sigma_{s,g} dW_s^{(i)}$. We observe the FHT of the processes $X^{(i)}$ for any $i = 1, \cdots, n$, i.e. $(\mathrm{T}_g^{X^{(1)}}, \dots, \mathrm{T}_g^{X^{(n)}})$.

We consider nonparametric estimation of the cdf and the pdf in a first step. The estimation is based on delta-sequences (see Equations (2.1) and (2.2), p. 176, Watson and Leadbetter (1964a)). The delta-sequences, i.e. a sequence of functions approaching the Dirac function asymptotically, are considered since they are general. They cover several types of smoothing estimation including the kernel estimation, orthogonal series estimation, Fourier transform estimation, and the histogram (see Walter and Blum (1979)). They are also suitable since the cdf estimator is proven to be uniformly convergent in Nadaraya (1964), and satisfying Chung-Smirnov property in Winter (1979). Reiss (1981) shows that the empirical estimator is asymptotically deficient compared to the smoothing estimator. We define the delta-sequences as δ_n for any $i = 1, \cdots, n$.

Following Section 4 of Watson and Leadbetter (1964b), we can then estimate the pdf and the cdf as

$$\widehat{f}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_n(t - \mathrm{T}_g^{X^{(i)}}) \text{ for } t \geq 0, \tag{5}$$

$$\widehat{F}_n(t) = \int_0^t \widehat{f}_n(u)du \text{ for } t \geq 0. \tag{6}$$

To estimate the hitting-time variance, we then plug the pdf and the cdf estimates into its formula (4). More specifically, an estimator of hitting-time variance is defined as

$$\widehat{\sigma}_{t,g}^{2,n} = \frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))}\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}} \quad \text{for any } t>0. \tag{7}$$

Since the asymptotic variance obtained in the CLT for estimation of hitting-time variance is based on the squared hitting-time variance $\sigma_{t,g}^4$, we study more deeply its estimation procedure. Thus, we consider estimation of the known deterministic function $k : \mathbb{R}_*^+ \to \mathbb{R}_*^+$ of the hitting-time variance. For instance, the most important case $k(x) = x^2$ corresponds to the estimation of the squared hitting-time variance. To estimate functions of hitting-time variance, we plug the estimator of hitting-time variance into the function. More specifically, the estimator for function of hitting-time variance is defined as

$$\widehat{k(\sigma_{t,g}^2)}^n = k(\widehat{\sigma}_{t,g}^{2,n})\mathbf{1}_{\{\widehat{\sigma}_{t,g}^{2,n}>0\}}. \tag{8}$$

We propose estimation for cumulative functions of hitting-time variance, i.e.

$$C_g(T_-,T_+) = \int_{T_-}^{T_+} k(\sigma_{t,g}^2)dt, \tag{9}$$

where T_- is the starting point and T_+ is the end point. This estimation is considered primarily to deduce an estimator of boundary, which is a tuning parameter. There are also of interest in their own right. When it is clear from the context, we will refer in what follows to $C_g = C_g(T_-,T_+)$. Since $\int_0^T \widehat{k(\sigma_{t,g}^2)}^n dt$ cannot be reexpressed in a more explicit form easily, we cannot estimate directly its formula as when estimating the cdf in Equation (6). We thus consider local estimation, based on empirical estimates for cumulative functions of hitting-time variance. We consider M_n intervals with equal length Δ_n, defined as

$$\Delta_n = \frac{T_+ - T_-}{M_n}. \tag{10}$$

Accordingly, we define the start and end points of the intervals as $T_0^n = T_-, T_1^n = T_- + \Delta_n, T_2^n = T_- + 2\Delta_n, \dots, T_{M_n}^n = T_+$. The estimator for cumulative functions of hitting-time variance is defined as

$$\widehat{C}_g^n = \sum_{l=0}^{M_n-1} \widehat{k(\sigma_{T_l^n,g}^2)}^n \Delta_n. \tag{11}$$

Finally, we propose estimation of the boundary g, which is a tuning parameter. This is based on estimation for cumulative functions of hitting-time variance. In the numerical simulation, we systematically find that the following choice gives good results. We set the boundary to $\overline{g}$ that normalizes the cumulative functions of hitting-time variance C_g, i.e. we define implicitly $\overline{g} > 0$ such that

$$\frac{C_{\overline{g}}(T_-,T_+)}{T_+ - T_-} = 1. \tag{12}$$

Accordingly, we define the boundary estimator implicitly as

$$\frac{\widehat{C}^n_{\widehat{g}_n}(T_-,T_+)}{T_+-T_-} = 1 \text{ if } \widehat{C}^n_1(T_-,T_+)>0, \tag{13}$$

$$\widehat{g}_n = 1 \text{ else.} \tag{14}$$

In Equation (12), we set the right side of the equation equal to unity. However, we can use a value different from unity.

4. Theory

In this section, we give the feasible CLT for estimation of hitting-time variance, functions of hitting-time variance, and cumulative function of hitting-time variance. We also show that the CLT for estimation of hitting-time variance and functions of hitting-time variance adapt when we estimate the boundary g.

We define the infimum time such that the cdf F is positive as K_f^0, i.e. $K_f^0 = \inf\{t>0 \text{ such that } F(t)>0\}$. We also define the infimum time such that F equals unity as K_f^1, i.e. $K_f^1 = \inf\{t>0 \text{ such that } F(t)=1\}$. For any function $h:\mathbb{R}^+\to\mathbb{R}^+$, $a\mapsto h(a)$, and any $A\subset\mathbb{R}^+$, we define the restriction of h to A as $h\restriction_A$ such that $h\restriction_A: A\to\mathbb{R}^+$, $a\mapsto h(a)$. For any $p\in\mathbb{R}$, $p\geq 1$ and $A\subset\mathbb{R}^+$ measurable, we define the set of p-integrable functions as

$$L_p(A) = \big\{h:A\to\mathbb{R}^+ \text{ measurable s.t. } \int_A |h(x)|^p \,\mathrm{d}x < +\infty\big\}.$$

For any $p\in\mathbb{R}$, $p\geq 1$, we define the set of locally p-integrable functions as

$$L_{p,\mathrm{loc}}(\mathbb{R}^+) = \big\{h:\mathbb{R}^+\to\mathbb{R}^+ \text{ measurable s.t. } h\restriction_K \in L_p(K)\ \forall K\subset\mathbb{R}^+,\ K \text{ compact}\big\}.$$

Let us give a set of assumptions required by Potiron (2023) (Assumption 4).

Assumption 1. We assume that there exists $\mu_f^0>0$ s.t. the hitting-time variance is locally integrable on $[K_f^0, K_f^0+\mu_f^0]$, i.e. $\sigma^2_{t,g}\restriction_{[K_f^0,K_f^0+\mu_f^0]} \in L_{1,loc}\big([K_f^0,K_f^0+\mu_f^0]\big)$. Moreover, we assume that K_f^1 is not finite.

Assumption 1 implies that $\sigma_g\in L_{2,\mathrm{loc}}(\mathbb{R}^+)$, thus X_t is a local martingale by Theorem I.4.40 (p. 48) in Jacod and Shiryaev (2003) with deterministic quadratic variation $\int_0^t \sigma^2_{u,g}du$. In particular, the assumption that K_f^1 is not finite is required since the hitting-time variance could explode when $t\to K_f^1$. All the examples we consider in the following of this paper satisfy Assumption 1.

We now give the assumptions on the delta-sequence. We introduce a sequence of positive functions $\delta_n:\mathbb{R}\to\mathbb{R}^+$ which will be called positive delta-sequence if it satisfies the following set of assumptions.

Assumption 2. We assume that

$$
\begin{aligned}
\int_{\mathbb{R}} \delta_n(t)dt &= 1 \text{ for } n \in \mathbb{N}, &(15)\\
\sup_{|t|\geq\lambda} \delta_n(t) &\to 0 \text{ for } \lambda > 0, &(16)\\
\int_{|t|\geq\lambda} \delta_n(t)dt &\to 0 \text{ as } n\to\infty \text{ for } \lambda > 0. &(17)
\end{aligned}
$$

For instance, we can consider the kernel function

$$\delta_n(t) = \frac{r(t/A_n)}{A_n \int_{\mathbb{R}} r(t)dt}. \tag{18}$$

Here $r : \mathbb{R} \to \mathbb{R}^+$ is bounded, integrable, and satisfies $r(t) = o(1/t)$ as $\mid t \mid\to \infty$ and A_n is the sequence of bandwidths which are positive and such that $A_n \to 0$ as $n \to \infty$. When compared to the general definition of delta-sequences given in Watson and Leadbetter (1964b) (Section 2, pp. 102-103), we restrict to the positive case since we want to avoid estimated negative pdf and cdf which would imply that the hazard function estimator (7) is not well-defined. This comes with a price of not allowing for kernel improving the rate of convergence of the bias and the MSE (Singpurwalla and Wong (1983)).

Moreover, we introduce the squared-average and $(2+\mu)$ power-average delta-sequence for $\mu > 0$ as

$$
\begin{aligned}
\alpha_n &= \int_{-\infty}^{\infty} \delta_n^2(t)dt,\\
\gamma_n &= \int_{-\infty}^{\infty} \delta_n(t)^{2+\mu}dt.
\end{aligned}
$$

We define the set of positive natural numbers as $\mathbb{N}^*$. Let us introduce another set of assumptions.

Assumption 3. We assume that

$$
\begin{aligned}
\alpha_n &< \infty \text{ for any } n \in \mathbb{N}^*, &(19)\\
\alpha_n &= o(n). &(20)
\end{aligned}
$$

We also assume that

$$
\begin{aligned}
\gamma_n &< \infty \text{ for any } n \in \mathbb{N}^*, &(21)\\
\frac{\gamma_n}{n^{\mu/2}\alpha_n^{1+\mu/2}} &\to 0. &(22)
\end{aligned}
$$

Finally, we assume that

$$
\begin{aligned}
\sqrt{\frac{n}{\alpha_n}}(\mathbb{E}[\widehat{f}_n(t)] - f(t)) &\to 0, &(23)\\
\sqrt{\frac{n}{\alpha_n}}(\mathbb{E}[\widehat{F}_n(t)] - F(t)) &\to 0. &(24)
\end{aligned}
$$

All the assumptions from Assumption 3 are required in Watson and Leadbetter (1964b). Expression (19) and Equation (20) are assumed for establishing the rate of convergence of the pdf estimator in Theorem 4 (p. 110), the CLT of the pdf estimator in Theorem 5 (p. 112) and the CLT of the ratio-based hazard estimator in Theorem 7 (p. 114). Expressions (21)-(22) are assumptions for Theorem 5 and Theorem 7. Expressions (23)-(24) are bias corrections required for Theorem 7.

Although Expressions (23)-(24) are a bit restrictive, there are many examples of distributions and delta-sequences from the literature on nonparametric estimation that satisfy them. For example, we introduce a real number $p > 1$ such that $\int_{\mathbb{R}} |t|^{p-1} |\Phi_f(t)| dt$, where $\Phi_f(t)$ is the characteristic function of f. We choose the kernel function (18) and assume that the Fourier transform $\Phi_r(t)$ is integrable on $\mathbb{R}$. Thus, we can deduce that $\alpha_n = A_n^{-1}$. We also assume that the limit of $(1 - \Phi_r(t))|t|^{1-q}$ as $t \to 0$ exists for some real number q such that $1 < q \leq p$. Then, it follows from Parzen (1962) and Watson and Leadbetter (1963) that

$$\sup_{t \in \mathbb{R}^+} \sqrt{\frac{n}{\alpha_n}} (\mathbb{E}[\widehat{f}_n(t)] - f(t)) \quad \to \quad 0, \tag{25}$$

as $n \to \infty$ provided we take $A_n = o(n^{1/(2q-1)})$. Hence, Expression (23) from Assumption 3 holds. It can be further shown provided $r(t) = O(|t|^{-q})$ that Expression (24) from Assumption 3 holds. This is obtained from Expression (25). Other examples are also possible. If the characteristic function $\Phi_f(t)$ tends to zero exponentially as $|t| \to \infty$, the above analysis holds.

As a consequence of Lemma 1 (p. 103) in Watson and Leadbetter (1964b) along with the fact that $\delta_n(t)$ is a positive delta-sequence, i.e. Assumption 2, and Expression (19), we can deduce that

$$\alpha_n \to \infty. \tag{26}$$

Finally, the use of Expressions (19)-(20) implies that

$$\sqrt{\frac{n}{\alpha_n}} \to \infty,$$

which is in fact the pdf estimator and ratio-based estimator rate of convergence.

We give now the feasible CLT for estimation of hitting-time variance. This CLT extends Theorem 7 from Watson and Leadbetter (1964b) when the denominator is equal to $f_g^W((P_g^W)^{-1}(F(t)))$ instead of $1 - F(t)$. We obtain $\sqrt{\frac{n}{\alpha_n}}$ as rate of convergence, which is equal to the rate of convergence in Theorem 7. We obtain the same numerator of the asymptotic variance, i.e. $f(t)$, but the denominator of the asymptotic variance is equal to $(f_g^W((P_g^W)^{-1}(F(t)))^2$ instead of $(1 - F(t))^2$. Moreover, we give consistent estimators of the asymptotic variance and feasible CLT, none of which are provided in Theorem 7. Our proofs also extend their machinery as a more complicated function form in the denominator implies deeper calculation. We require the positivity assumption on f which implies that $0 < F(t) < 1$ and $\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}} \xrightarrow{\mathbb{P}} 1$. We also require the continuity of f at the time t, which are also assumptions in Theorem 7 from Watson and Leadbetter (1964b). If the density f is not continuous at the time t, its estimator $\widehat{f}_n(t)$ is not consistent. Thus, the hitting-time variance estimator $\widehat{\sigma}^{2,n}_{t,g}$ is not even consistent in that case.

Theorem 1. *We assume that Assumptions 1, 2 and 3 hold. We have for any $t > 0$, assuming that it is a continuity point of f and $f(t) > 0$, the standard CLT*

$$\sqrt{\frac{n}{\alpha_n}}(\widehat{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g}) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{f(t)}{f^W_g((P^W_g)^{-1}(F(t)))^2}\Big) \tag{27}$$

as $n \to \infty$. We also obtain the feasible normalized CLT

$$\sqrt{\frac{n}{\alpha_n}\frac{f^W_g((P^W_g)^{-1}(\widehat{F}_n(t)))^2}{\widehat{f}_n(t)}}(\widehat{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g}) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{28}$$

By Equation (4), we can rewrite the asymptotic variance as

$$\frac{f(t)}{f^W_g((P^W_g)^{-1}(F(t)))^2} = \frac{\sigma^4_{t,g}}{f(t)}.$$

Then, we can show that the asymptotic variance diverges, i.e $\frac{\sigma^4_{t,g}}{f(t)} \to \infty$, as $t \to 0$ or $t \to \infty$. The same problem appears when estimating the standard hazard function in Watson and Leadbetter (1964b) (Theorem 7).

For any $A \subset \mathbb{R}^+$ and $B \subset \mathbb{R}^+$, we denote the space $\mathcal{C}_k$ of functions $h : A \to B$ which are continuously differentiable k times as $\mathcal{C}_k(A, B)$. To estimate the functions of hitting-time variance, we make the following smoothness assumptions on k.

Assumption 4. We assume that there exists $\eta > 0$ s.t.

$$k \in \mathcal{C}_2([\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta], \mathbb{R}^+_*), \tag{29}$$

$$k \in \mathcal{C}_3([\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta], \mathbb{R}^+_*). \tag{30}$$

We now give some discussion about Assumption 4. First, we have that k relies on the hitting-time variance $\sigma^2_{t,g}$. Thus, the assumption on k depends on the time t. However, this is a local assumption required for the CLT of the hitting-time variance function at the time t. We emphasize on the fact that the function k does not depend on the time t. Moreover, the hitting-time variance estimator $\widehat{\sigma}^{2,n}_{t,g}$ is not necessarily included in the domain $[\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta]$ of the function k. However, we can deduce by Theorem 1 that $\mathbb{P}(\widehat{\sigma}^{2,n}_{t,g} \in [\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta]) \to 1$ as $n \to \infty$ and this is sufficient for our CLT. Finally, we consider two distinct assumptions, i.e. Expressions (29) and (30). We have by definition that Expression (29) is weaker than Expression (30). The reason why we introduce two distinct assumptions is that Expression (29) is used for the standard CLT while Expression (30) is required for the feasible normalized CLT.

We give now the CLT of the hitting-time variance function estimator. This CLT extends Theorem 7 from Watson and Leadbetter (1964b) as we consider estimation of functions of the hitting-time variance instead of estimation of the standard hazard function. It also slightly extends our Theorem 1 as we consider estimation of the hitting-time variance function instead of estimation of the hitting-time variance. We obtain $\sqrt{\frac{n}{\alpha_n}}$ as rate of convergence, which is equal to the rate of convergence in Theorem 7 and our Theorem 1. We obtain that the asymptotic variance is equal to $\frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}$ instead of $\frac{f(t)}{(1-F(t))^2}$ in Theorem 7 and $\frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}$ in our Theorem 1. Moreover, we give consistent estimators of the asymptotic variance and feasible CLT, none of which are provided in Theorem 7. Our proofs also extend their machinery and slightly our proofs of Theorem 1 as the presence of the function implies the use of a Taylor expansion along with Assumption 4.

Theorem 2. *We assume that Assumptions 1, 2 and 3 hold. We have for any $t > 0$, assuming that it is a continuity point of f with $f(t) > 0$ and that Assumption 4 Expression (29) holds, the standard CLT*

$$\sqrt{\frac{n}{\alpha_n}}(\widehat{k(\sigma_{t,g}^2)}^n - k(\sigma_{t,g}^2)) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}\Big) \tag{31}$$

as $n \to \infty$. If we also assume that Assumption 4 Expression (30) holds, we obtain the feasible normalized CLT

$$\sqrt{\frac{n}{\alpha_n}\frac{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))^2}{(\widehat{k'(\sigma_{t,g}^2)}^n)^2 \widehat{f}_n(t)}}(\widehat{k(\sigma_{t,g}^2)}^n - k(\sigma_{t,g}^2)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{32}$$

We can also show that the asymptotic variance diverges as $t \to 0$ or $t \to \infty$.

To estimate the cumulative functions of hitting-time variance, we make another set of assumptions.

Assumption 5. We assume that

$$\begin{aligned} M_n &\to \infty \text{ such that } \sqrt{\frac{n}{\alpha_n}} = o(M_n) \text{ as } n \to \infty, && (33)\\ 0 &< T_- < T_+ < \infty, && (34)\\ f \restriction_{[T_-,T_+]} &\in \mathcal{C}_1([T_-,T_+],\mathbb{R}^+), && (35)\\ f(t) &> 0 \text{ for any } t \in [T_-,T_+]. && (36) \end{aligned}$$

We also assume that

$$\begin{aligned} \sqrt{\frac{n}{\alpha_n}}\sup_{T_-\leq t\leq T_+}\big|\mathbb{E}[\widehat{f}_n(t)] - f(t)\big| &\to 0, && (37)\\ \sqrt{\frac{n}{\alpha_n}}\sup_{T_-\leq t\leq T_+}\big|\mathbb{E}[\widehat{F}_n(t)] - F(t)\big| &\to 0. && (38) \end{aligned}$$

Finally, we assume that there exists $\eta > 0$ s.t.

$$k \in \mathcal{C}_2\Big(\Big[\inf_{T_- \le t \le T_+} \sigma^2_{t,g} - \eta, \sup_{T_- \le t \le T_+} \sigma^2_{t,g} + \eta\Big], \mathbb{R}\Big), \tag{39}$$

$$k \in \mathcal{C}_3\Big(\Big[\inf_{T_- \le t \le T_+} \sigma^2_{t,g} - \eta, \sup_{T_- \le t \le T_+} \sigma^2_{t,g} + \eta\Big], \mathbb{R}\Big). \tag{40}$$

Assumption 5 Expression (33) is required for local estimation. Assumption 5 Expression (34), i.e. $T_- \neq 0$ and $T_+ \neq \infty$, is required since the asymptotic variance in the CLT (45) diverges as $t \to 0$ or $t \to \infty$. Assumption 5 Expression (35) is required for the use of a Taylor expansion. Assumption 5 Expression (36), i.e. uniform positivity of f, is required since we use locally Theorem 2. Assumption 5 Expressions (37)-(38) are uniform bias corrections. Assumption 5 Expressions (39)-(40) are uniform function smoothness assumptions.

Although Expressions (37)-(38) are a bit restrictive, there are many examples of distributions and delta-sequences from the literature on nonparametric estimation that satisfy them. There are directly implied by Expression (25) for the specific example introduced in the discussion.

We now give some discussion about Expressions (39)-(40) from Assumption 5. First, we have that Expression (39) is stronger than Expression (29) from Assumption 4. We also have that Expression (39) is stronger than Expression (29) from Assumption 4. Then, we have that k relies on the hitting-time variance $\sigma^2_{t,g}$ for any time $t \in [T_-, T_+]$. Thus, the assumption on k does not depend locally on the time t. First, we have that Expression (39) is stronger than Expression (29) from Assumption 4. Moreover, the hitting-time variance estimator $\widehat{\sigma}^{2,n}_{t,g}$ is not necessarily included in the domain $\big[\inf_{T_- \le t \le T_+} \sigma^2_{t,g} - \eta, \sup_{T_- \le t \le T_+} \sigma^2_{t,g} + \eta\big]$ of the function k. However, we can deduce by an extension of Theorem 2 that $\mathbb{P}(\widehat{\sigma}^{2,n}_{t,g} \in \big[\inf_{T_- \le t \le T_+} \sigma^2_{t,g} - \eta, \sup_{T_- \le t \le T_+} \sigma^2_{t,g} + \eta\big]) \to 1$ as $n \to \infty$ and this is sufficient for our CLT. Finally, we consider two distinct assumptions, i.e. Expressions (39) and (40). We have by definition that Expression (39) is weaker than Expression (40). The reason why we introduce two distinct assumptions is that Expression (39) is used for the standard CLT while Expression (40) is required for the feasible normalized CLT.

We give now the CLT for the cumulative function of hitting-time variance estimator. The CLT results are obtained primarily to deduce a boundary tuning parameter estimator, but are also of interest in their own right. This CLT extends Theorem 7 from Watson and Leadbetter (1964b) as we consider estimation of cumulative functions of the hitting-time variance instead of estimation of the standard hazard function. It also extends our Theorem 1 and Theorem 2 as we consider estimation of cumulative functions of the hitting-time variance instead of estimation of functions of the hitting-time variance. We obtain $\sqrt{\frac{n}{\alpha_n}}$ as rate of convergence, which is equal to the rate of convergence in Theorem 7 and our Theorems 1-2. We obtain the asymptotic variance equal to the cumulative asymptotic variance in the CLT (45), i.e. $\int_{T_-}^{T_+} \frac{k'(\sigma^2_{t,g})^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt$, instead of $\frac{f(t)}{(1-F(t))^2}$ in Theorem 7 and $\frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}$ in our Theorem 2. Moreover, we give consistent estimators of the asymptotic variance and feasible CLT, none of which are provided in Theorem 7. Our proofs also extend their machinery and our proofs of Theorem 1 and Theorem 2 as cumulative estimation requires uniform arguments on $[T_-, T_+]$.

Theorem 3. *We assume that Assumptions 1, 2, 3 and Assumption 5 Expressions (33),(34),(35),(36),(37) and (38) hold. We have, assuming that Assumption 5 Expression (39) holds, the standard CLT*

$$\sqrt{\frac{n}{\alpha_n}}(\widehat{C}_g^n - C_g) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, (T_+ - T_-)\int_{T_-}^{T_+} \frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}dt\Big) \tag{41}$$

as $n \to \infty$. If we also assume that Assumption 5 Expression (40) holds, we obtain the feasible normalized CLT

$$\sqrt{\frac{n}{\alpha_n}\Big((T_+ - T_-)\sum_{l=0}^{M_n-1}\frac{(k'(\widehat{\sigma_{T_l^n,f}^2})^n)^2\widehat{f}_n(T_l^n)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(T_l^n)))^2}\Delta_n\Big)^{-1}}(\widehat{C}_g^n - C_g) \tag{42}$$
$$\xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

We can also show that the asymptotic variance diverges as $T_- \to 0$ or $T_+ \to \infty$.

In what follows, we adapt Theorem 1 and Theorem 2 when the boundary g is estimated. First, we note that the asymptotic variance in the CLT of Theorem 1 and Theorem 2 when normalized respectively by the hitting-time variance and the function of hitting-time variance does not depend on the boundary g, i.e. estimators based on different values of g are asymptotically equivalent. Indeed, we can reexpress the CLT (27) in Theorem 1 when normalized by the hitting-time variance as

$$\sqrt{\frac{n}{\alpha_n}}\frac{\widehat{\sigma}_{t,g}^{2,n} - \sigma_{t,g}^2}{\sigma_{t,g}^2} \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{1}{f(t)}\Big). \tag{43}$$

Here, we use Equation (4) and the fact that $\sqrt{a}\mathcal{N}(0,b) = \mathcal{N}(0,ab)$ for any $a > 0$ and $b > 0$. Moreover, we can use the same arguments in the other CLTs of Theorem 1 and Theorem 2. Thus, there is no theoretical asymptotic gain by choosing a particular g.

The existence and uniqueness of $\overline{g}$ and $\widehat{g}_n$, as well as the consistency of $\widehat{g}_n$ to $\overline{g}$, will be shown in the following proposition. We make the following assumption on k for that.

Assumption 6. We assume that k is a strictly monotone function.

The proof of Proposition 1 extends the arguments in the proof of Theorem 3 since boundary tuning parameter estimation requires uniform arguments on g.

Proposition 1. *We assume that Assumptions 1 and 6 hold and that $0 < T_- < T_+$. Then, there exists a unique $\overline{g} > 0$ which satisfies Equation (12) and a unique*

$\widehat{g}_n$ which satisfies Equations (13) and (14). If we also assume that Assumptions 2,3 and Assumption 5 Expressions (33),(34),(35),(36),(37),(38) and (39) hold, then we have the consistency of the boundary tuning parameter, i.e.

$$\widehat{g}_n \xrightarrow{\mathbb{P}} \overline{g}. \tag{44}$$

Finally, the adaptation of Theorem 1 and Theorem 2 when the boundary tuning parameter is estimated will be shown in the next corollary. The asymptotic variance is unchanged by the estimation of the bounding tuning parameter.

Corollary 1. *We assume that Assumptions 1, 2, 3, 6 and Assumption 5 Expressions (33),(34),(35),(36),(37) and (38) hold and that $0 < T_{-} < t \leq T_{+}$. We have, assuming that Assumption 5 Expression (39) holds, the standard CLT*

$$\sqrt{\frac{n}{\alpha_n}}(k(\widehat{\sigma^2_{t,\widehat{g}_n}})^n - k(\sigma^2_{t,\overline{g}})) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{k'(\sigma^2_{t,\overline{g}})^2 f(t)}{f^W_{\overline{g}}((P^W_{\overline{g}})^{-1}(F(t)))^2}\Big) \tag{45}$$

as $n \to \infty$. If we also assume that Assumption 5 Expression (40) holds, we obtain the feasible normalized CLT

$$\sqrt{\frac{n}{\alpha_n}\frac{f^W_{\widehat{g}_n}((P^W_{\widehat{g}_n})^{-1}(\widehat{F}_n(t)))^2}{(k'(\widehat{\sigma^2_{t,\widehat{g}_n}})^n)^2 \widehat{f}_n(t)}}(k(\widehat{\sigma^2_{t,\widehat{g}_n}})^n - k(\sigma^2_{t,\overline{g}})) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{46}$$

5. Discussion

In this section, we discuss about an appropriate choice of the delta-sequence, α_n, T_{-}, T_{+} and M_n in practice. We also discuss about the case of censored data.

First, we recommend to use the triangular kernel as delta-sequence since it is the nonnegative kernel with the smallest MSE in Singpurwalla and Wong (1983) (Table 1). Since its support is bounded, it also prevents from obtaining too many times positive estimation for which the pdf is null.

Second, Müller and Wang (1990) give a general principle to obtain local bandwidth, i.e. minimizing the MSE in case of hazard estimation based on ordered sample values. Lo, Mack and Wang (1989) give the MSE in case of estimation for standard hazard function, but they do not minimize it with respect to the bandwidth. Moreover, since the asymptotic variance diverges at the extremities, i.e. as $t \to 0$ or $t \to \infty$, and since the support of the delta-sequence could exceed the available range of data, there are bias problems at the extremities. The same problem appears for estimation of standard hazard function and is referred as extremity effects. Müller and Wang (1994) propose standard hazard function estimation with varying kernels and bandwidths in their Equation (2.2) (p. 62), based on Gasser and Müller (1979) earlier proposal. In their algorithm, they recommend to start with $b = \frac{1}{8n^{1/5}}$, which can be expressed in terms of our tuning parameter as

$$\alpha_n = 8n^{1/5}. \tag{47}$$

With this choice, Assumption 3 holds. A rigorous study of the hitting-time variance MSE and the choice of local efficient bandwidth and delta-sequence is above the scope of this paper and left for future work. However, we will find in our numerical simulation that a larger tuning parameter value, i.e.

$$\alpha_n = 8n^{\frac{9}{10}}, \tag{48}$$

will give better results. With this choice, we obviously have that $\alpha_n = o(n)$ and thus Equation (20) from Assumption 3 is satisfied.

Finally, we recommend to choose T_- and T_+ such that $[T_-, T_+]$ covers around 90% of the cdf. We also recommend to set the number of intervals such that $M_n = \left(\frac{n}{\alpha_n}\right)^{\frac{1}{4}}$. With this choice, we have that Assumption 5 Expression (33) is satisfied.

Since our empirical application is based on financial data, we do observe the data fully. However, the case of censored data may be of interest for applications in survival analysis and medicine. Extensions of the ratio hazard estimator to censored data are available in Blum and Susarla (1980), Földes, Rejto and Winter (1981) and Lo, Mack and Wang (1989). They are based on Kaplan-Meier estimation (see Kaplan and Meier (1958)). Thus, we recommend to use the Kaplan-Meier estimator, rather than the empirical estimator. Then, we can use delta-sequences based on the Kaplan-Meier estimator. We conjecture that the theory holds. However, investigating this adaptation is above the scope of this paper and left for future work.

6. Numerical simulation

In this section, we give a numerical simulation to assess the CLT of the infeasible and feasible Z-statistics for estimation of hitting-time variance, i.e.

$$\begin{aligned} Z_t^{\sigma_g^2, inf} &= \sqrt{\frac{n}{\alpha_n} \frac{f_{\overline{g}}^W((P_{\overline{g}}^W)^{-1}(F(t)))^2}{f(t)}} (\widehat{\sigma}_{t,\overline{g}}^{2,n} - \sigma_{t,\overline{g}}^2), \\ Z_t^{\sigma_g^2, feas} &= \sqrt{\frac{n}{\alpha_n} \frac{f_{\widehat{g}_n}^W((P_{\widehat{g}_n}^W)^{-1}(\widehat{F}_n(t)))^2}{\widehat{f}_n(t)}} (\widehat{\sigma}_{t,\widehat{g}_n}^{2,n} - \sigma_{t,\overline{g}}^2). \end{aligned}$$

We also assess the CLT of the infeasible and feasible Z-statistics to estimate the functions of hitting-time variance, i.e.

$$\begin{aligned} Z_t^{k(\sigma_g^2), inf} &:= \sqrt{\frac{n}{\alpha_n} \frac{f_{\overline{g}}^W((P_{\overline{g}}^W)^{-1}(F(t)))^2}{(k'(\sigma_{t,\overline{g}}^2))^2 f(t)}} (k\widehat{(\sigma_{t,\overline{g}}^2)}^n - k(\sigma_{t,\overline{g}}^2)), \\ Z_t^{k(\sigma_g^2), feas} &:= \sqrt{\frac{n}{\alpha_n} \frac{f_{\widehat{g}_n}^W((P_{\widehat{g}_n}^W)^{-1}(\widehat{F}_n(t)))^2}{(k'\widehat{(\sigma_{t,\widehat{g}_n}^2)}^n)^2 \widehat{f}_n(t)}} (k\widehat{(\sigma_{t,\widehat{g}_n}^2)}^n - k(\sigma_{t,\overline{g}}^2)). \end{aligned}$$

We consider the Levy distribution since it is a right skewed distribution with fat tail, which is what we see in our empirical application. We set the scale parameter $c = 1.4 \times 10^{-5}$ which is the value obtained in our empirical application and the boundary equal to the square root of the parameter, i.e. $g = \sqrt{c}$, which makes the hitting-time variance constant equal to unity, i.e. $\sigma_t^f = 1$ for any $t > 0$ (see Appendix 9.1 in the supplementary materials). We also consider the case $c = 5 \times 10^{-4}$ where the stochastic process is far from the boundary.

As in Watson and Leadbetter (1964b) (see Section 5, Equation 5.2, p. 115), we consider the triangle kernel function defined as for $\delta > 0$

$$\begin{aligned}\delta_n(t) &= \delta(1-\delta|t|) \text{ if } |t| \leq \frac{1}{\delta},\\ \delta_n(t) &= 0 \text{ else.}\end{aligned}$$

With that choice, we have that $\alpha_n = \int_{-\infty}^{\infty} \delta_n^2(t)dt = \frac{2\delta}{3}$. For large n, we set $\alpha_n = 8n^{\frac{9}{10}}$ when $10^{-7} \leq t < 10^{-3}$, $\alpha_n = 8n^{\frac{1}{2}}$ when $10^{-3} \leq t < 10^{-1}$, and $\alpha_n = 8n^{\frac{1}{4}}$ when $10^{-1} \leq t < 10$. For small n, we set $\alpha_n = 8n^2$ when $10^{-7} \leq t < 10^{-3}$, and $\alpha_n = 8n$ when $10^{-3} \leq t < 10^0$.

We consider the number of iterations $n = 1,000,000, 2,000,000, 5,000,000, 10,000,000$, which are large values but adequate since they are smaller than the value used in our empirical application. We also explore the cases of small $n = 1,000$ and intermediary $n = 50,000$ for applications in survival analysis and medicine. We also use $S = 5,000,000$ which corresponds to the number of partitions in the numerical discretization of the integral when implementing the cdf estimator (6). Moreover, the number of iterations is fixed to $N = 1000$. Finally, we set $T_- = 5\times10^{-6}$ and $T_+ = 5\times10^{-4}$, which covers around 90% of the distribution. We also set the number of intervals such that $M_n = \left(\frac{n}{\alpha_n}\right)^{\frac{1}{4}}$. We choose the squared function, i.e. $k(x) = x^2$, which corresponds to the estimation of the squared hitting-time variance.

Table 1: Summary statistics for Levy distribution at time $t = 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled ”5%” and ”95%” respectively correspond to the 5%-quantile and 95%-quantile.

c		1.4×10^{-5}				5×10^{-4}			
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	-0.004	1.000	-1.647	1.710	0.041	1.010	-1.647	1.688
	$Z_t^{\sigma_g^2,feas}$	-0.023	1.002	-1.704	1.651	0.022	1.010	-1.705	1.635
	$Z_t^{k(\sigma_g^2),inf}$	0.015	1.000	-1.595	1.766	0.061	1.012	-1.594	1.743
	$Z_t^{k(\sigma_g^2),feas}$	-0.043	1.006	-1.761	1.601	0.002	1.013	-1.763	1.584
2,000,000	$Z_t^{\sigma_g^2,inf}$	-0.004	1.007	-1.589	1.662	-0.038	1.013	-1.768	1.618

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Table 1: Summary statistics for Levy distribution at time $t = 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c		1.4×10^{-5}				5×10^{-4}			
n	Statistics	Mean	Std	5.0%	95%	Mean	Std	5.0%	95%
	$Z_t^{\sigma_g^2,feas}$	-0.020	1.010	-1.633	1.621	-0.055	1.015	-1.824	1.578
	$Z_t^{k(\sigma_g^2),inf}$	0.012	1.006	-1.548	1.706	-0.021	1.012	-1.717	1.661
	$Z_t^{k(\sigma_g^2),feas}$	-0.037	1.015	-1.677	1.580	-0.072	1.019	-1.881	1.538
5,000,000	$Z_t^{\sigma_g^2,inf}$	0.020	0.961	-1.602	1.574	0.016	1.037	-1.601	1.709
	$Z_t^{\sigma_g^2,feas}$	0.008	0.961	-1.637	1.542	0.002	1.035	-1.633	1.673
	$Z_t^{k(\sigma_g^2),inf}$	0.031	0.961	-1.569	1.606	0.030	1.040	-1.568	1.747
	$Z_t^{k(\sigma_g^2),feas}$	-0.004	0.963	-1.672	1.512	-0.012	1.035	-1.669	1.637
10,000,000	$Z_t^{\sigma_g^2,inf}$	0.008	1.047	-1.764	1.680	0.027	1.003	-1.615	1.659
	$Z_t^{\sigma_g^2,feas}$	-0.004	1.047	-1.797	1.649	0.016	1.004	-1.644	1.630
	$Z_t^{k(\sigma_g^2),inf}$	0.020	1.047	-1.731	1.711	0.038	1.003	-1.586	1.690
	$Z_t^{k(\sigma_g^2),feas}$	-0.016	1.048	-1.833	1.621	0.005	1.005	-1.674	1.601

Table 2: Summary statistics for Levy distribution at time $t = 2.5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 2.5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c		1.4×10^{-5}				5×10^{-4}			
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	-0.009	1.002	-1.601	1.702	0.025	1.009	-1.672	1.707
	$Z_t^{\sigma_g^2,feas}$	-0.041	1.008	-1.684	1.615	-0.012	1.013	-1.784	1.615
	$Z_t^{k(\sigma_g^2),inf}$	0.022	1.003	-1.522	1.791	0.061	1.014	-1.571	1.812
	$Z_t^{k(\sigma_g^2),feas}$	-0.072	1.021	-1.776	1.539	-0.049	1.024	-1.906	1.526
2,000,000	$Z_t^{\sigma_g^2,inf}$	0.017	0.999	-1.664	1.597	0.019	1.003	-1.619	1.707
	$Z_t^{\sigma_g^2,feas}$	-0.010	1.001	-1.740	1.525	-0.011	1.007	-1.709	1.626
	$Z_t^{k(\sigma_g^2),inf}$	0.042	1.001	-1.592	1.663	0.050	1.005	-1.540	1.795
	$Z_t^{k(\sigma_g^2),feas}$	-0.036	1.007	-1.822	1.467	-0.042	1.017	-1.802	1.550

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Table 2: Summary statistics for Levy distribution at time $t = 2.5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 2.5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5.0%	95%	Mean	Std	5.0%	95%
5,000,000	$Z_t^{\sigma_g^2,inf}$	-0.028	1.034	-1.768	1.609	-0.012	0.960	-1.581	1.629
	$Z_t^{\sigma_g^2,feas}$	-0.050	1.040	-1.833	1.557	-0.034	0.962	-1.643	1.568
	$Z_t^{k(\sigma_g^2),inf}$	-0.006	1.032	-1.704	1.662	0.010	0.961	-1.521	1.692
	$Z_t^{k(\sigma_g^2),feas}$	-0.073	1.050	-1.905	1.509	-0.057	0.968	-1.710	1.511
10,000,000	$Z_t^{\sigma_g^2,inf}$	-0.029	0.976	-1.601	1.590	0.009	1.014	-1.663	1.606
	$Z_t^{\sigma_g^2,feas}$	-0.045	0.979	-1.645	1.546	-0.012	1.016	-1.721	1.556
	$Z_t^{k(\sigma_g^2),inf}$	-0.012	0.976	-1.556	1.633	0.030	1.014	-1.608	1.658
	$Z_t^{k(\sigma_g^2),feas}$	-0.062	0.983	-1.693	1.506	-0.033	1.021	-1.783	1.509

Table 3: Summary statistics for Levy distribution at time $t = 5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	-0.013	1.028	-1.682	1.717	0.002	0.987	-1.555	1.687
	$Z_t^{\sigma_g^2,feas}$	-0.065	1.037	-1.851	1.590	-0.056	0.993	-1.722	1.540
	$Z_t^{k(\sigma_g^2),inf}$	0.038	1.034	-1.546	1.859	0.059	1.003	-1.412	1.855
	$Z_t^{k(\sigma_g^2),feas}$	-0.118	1.063	-2.030	1.478	-0.116	1.023	-1.915	1.412
2,000,000	$Z_t^{\sigma_g^2,inf}$	0.031	1.029	-1.625	1.694	0.040	0.991	-1.462	1.739
	$Z_t^{\sigma_g^2,feas}$	-0.013	1.035	-1.753	1.585	-0.009	0.992	-1.584	1.606
	$Z_t^{k(\sigma_g^2),inf}$	0.074	1.035	-1.518	1.810	0.088	1.006	-1.356	1.889
	$Z_t^{k(\sigma_g^2),feas}$	-0.057	1.055	-1.887	1.489	-0.058	1.009	-1.719	1.488
5,000,000	$Z_t^{\sigma_g^2,inf}$	0.005	1.011	-1.688	1.708	0.024	0.958	-1.540	1.608
	$Z_t^{\sigma_g^2,feas}$	-0.028	1.014	-1.789	1.618	-0.013	0.961	-1.641	1.515
	$Z_t^{k(\sigma_g^2),inf}$	0.038	1.015	-1.596	1.802	0.060	0.964	-1.446	1.709

Continued on next page

Table 3: Summary statistics for Levy distribution at time $t = 5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5.0%	95%	Mean	Std	5.0%	95%
	$Z_t^{k(\sigma_g^2),feas}$	-0.062	1.024	-1.898	1.538	-0.049	0.973	-1.755	1.430
10,000,000	$Z_t^{\sigma_g^2,inf}$	-0.045	0.970	-1.580	1.559	0.030	0.982	-1.559	1.649
	$Z_t^{\sigma_g^2,feas}$	-0.070	0.975	-1.653	1.495	-0.003	0.987	-1.648	1.564
	$Z_t^{k(\sigma_g^2),inf}$	-0.019	0.969	-1.512	1.625	0.062	0.985	-1.479	1.739
	$Z_t^{k(\sigma_g^2),feas}$	-0.097	0.985	-1.730	1.437	-0.035	0.998	-1.742	1.487

Table 4: Summary statistics for Levy distribution at time $t = 7.5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 7.5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	0.029	1.008	-1.563	1.742	0.028	0.983	-1.485	1.713
	$Z_t^{\sigma_g^2,feas}$	-0.037	1.018	-1.751	1.575	-0.049	0.992	-1.699	1.526
	$Z_t^{k(\sigma_g^2),inf}$	0.094	1.028	-1.407	1.936	0.105	1.011	-1.311	1.946
	$Z_t^{k(\sigma_g^2),feas}$	-0.105	1.060	-1.969	1.432	-0.130	1.042	-1.960	1.362
2,000,000	$Z_t^{\sigma_g^2,inf}$	-0.056	0.959	-1.684	1.526	-0.015	0.985	-1.572	1.626
	$Z_t^{\sigma_g^2,feas}$	-0.108	0.977	-1.862	1.413	-0.081	0.998	-1.770	1.474
	$Z_t^{k(\sigma_g^2),inf}$	-0.006	0.957	-1.531	1.651	0.049	1.001	-1.407	1.802
	$Z_t^{k(\sigma_g^2),feas}$	-0.161	1.015	-2.068	1.313	-0.150	1.046	-2.005	1.343
5,000,000	$Z_t^{\sigma_g^2,inf}$	-0.054	0.985	-1.695	1.629	-0.030	0.980	-1.566	1.617
	$Z_t^{\sigma_g^2,feas}$	-0.096	0.991	-1.829	1.523	-0.082	0.989	-1.712	1.495
	$Z_t^{k(\sigma_g^2),inf}$	-0.013	0.989	-1.572	1.742	0.021	0.987	-1.436	1.756
	$Z_t^{k(\sigma_g^2),feas}$	-0.140	1.007	-1.984	1.430	-0.135	1.015	-1.883	1.386
10,000,000	$Z_t^{\sigma_g^2,inf}$	-0.060	1.048	-1.798	1.653	0.002	0.994	-1.549	1.744
	$Z_t^{\sigma_g^2,feas}$	-0.100	1.057	-1.925	1.563	-0.042	0.998	-1.668	1.624

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Table 4: Summary statistics for Levy distribution at time $t = 7.5 \times 10^{-5}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 7.5 \times 10^{-3}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5.0%	95%	Mean	Std	5.0%	95%
	$Z_t^{k(\sigma_g^2),inf}$	-0.020	1.048	-1.682	1.752	0.046	1.003	-1.442	1.879
	$Z_t^{k(\sigma_g^2),feas}$	-0.141	1.076	-2.068	1.480	-0.087	1.016	-1.802	1.514

Table 5: Summary statistics for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 10^{-2}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c			1.4×10^{-5}				5×10^{-4}		
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	0.022	1.028	-1.566	1.753	0.003	0.997	-1.608	1.663
	$Z_t^{\sigma_g^2,feas}$	-0.064	1.047	-1.805	1.542	-0.099	1.036	-1.936	1.446
	$Z_t^{k(\sigma_g^2),inf}$	0.105	1.054	-1.374	1.994	0.101	1.031	-1.355	1.934
	$Z_t^{k(\sigma_g^2),feas}$	-0.154	1.116	-2.099	1.378	-0.215	1.169	-2.379	1.268
2,000,000	$Z_t^{\sigma_g^2,inf}$	0.059	1.004	-1.533	1.759	-0.032	0.994	-1.579	1.611
	$Z_t^{\sigma_g^2,feas}$	-0.008	1.008	-1.723	1.596	-0.116	1.018	-1.838	1.434
	$Z_t^{k(\sigma_g^2),inf}$	0.126	1.027	-1.378	1.962	0.049	1.012	-1.374	1.824
	$Z_t^{k(\sigma_g^2),feas}$	-0.076	1.041	-1.944	1.445	-0.208	1.089	-2.161	1.284
5,000,000	$Z_t^{\sigma_g^2,inf}$	0.028	1.022	-1.598	1.809	-0.058	0.962	-1.626	1.479
	$Z_t^{\sigma_g^2,feas}$	-0.027	1.026	-1.752	1.660	-0.120	0.981	-1.835	1.355
	$Z_t^{k(\sigma_g^2),inf}$	0.083	1.035	-1.464	1.980	0.003	0.967	-1.453	1.623
	$Z_t^{k(\sigma_g^2),feas}$	-0.083	1.049	-1.928	1.528	-0.187	1.027	-2.083	1.245
10,000,000	$Z_t^{\sigma_g^2,inf}$	0.039	0.998	-1.591	1.765	-0.025	0.983	-1.575	1.633
	$Z_t^{\sigma_g^2,feas}$	-0.005	0.999	-1.719	1.641	-0.080	1.004	-1.732	1.507
	$Z_t^{k(\sigma_g^2),inf}$	0.083	1.009	-1.480	1.902	0.028	0.981	-1.439	1.780
	$Z_t^{k(\sigma_g^2),feas}$	-0.049	1.012	-1.860	1.531	-0.139	1.050	-1.914	1.392

Table 6: Summary statistics for Levy distribution at time $t = 2.5 \times 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$ and at time $t = 2.5 \times 10^{-2}$ for the case $c = 5 \times 10^{-4}$. This is based on $N = 1,000$ iterations. The columns entitled "5%" and "95%" respectively correspond to the 5%-quantile and 95%-quantile.

c		1.4×10^{-5}				5×10^{-4}			
n	Statistics	Mean	Std	5%	95%	Mean	Std	5%	95%
1,000,000	$Z_t^{\sigma_g^2,inf}$	0.102	0.999	-1.497	1.775	0.011	1.033	-1.456	1.817
	$Z_t^{\sigma_g^2,feas}$	-0.063	1.095	-2.052	1.426	-0.221	1.178	-2.178	1.397
	$Z_t^{k(\sigma_g^2),inf}$	0.256	1.083	-1.155	2.256	0.217	1.226	-1.047	2.455
	$Z_t^{k(\sigma_g^2),feas}$	-0.311	2.256	-2.912	1.174	-0.645	2.631	-3.543	1.107
2,000,000	$Z_t^{\sigma_g^2,inf}$	0.023	0.989	-1.469	1.715	-0.014	0.960	-1.462	1.596
	$Z_t^{\sigma_g^2,feas}$	-0.107	1.026	-1.866	1.428	-0.182	1.054	-2.018	1.297
	$Z_t^{k(\sigma_g^2),inf}$	0.149	1.059	-1.192	2.092	0.136	1.031	-1.115	2.010
	$Z_t^{k(\sigma_g^2),feas}$	-0.261	1.212	-2.428	1.210	-0.416	1.465	-2.934	1.076
5,000,000	$Z_t^{\sigma_g^2,inf}$	0.047	1.027	-1.571	1.840	0.024	0.994	-1.502	1.739
	$Z_t^{\sigma_g^2,feas}$	-0.064	1.058	-1.906	1.569	-0.109	1.028	-1.922	1.444
	$Z_t^{k(\sigma_g^2),inf}$	0.155	1.075	-1.319	2.185	0.151	1.062	-1.211	2.130
	$Z_t^{k(\sigma_g^2),feas}$	-0.189	1.194	-2.355	1.355	-0.263	1.192	-2.531	1.221
10,000,000	$Z_t^{\sigma_g^2,inf}$	-0.038	1.010	-1.620	1.710	0.036	1.000	-1.508	1.753
	$Z_t^{\sigma_g^2,feas}$	-0.130	1.045	-1.904	1.498	-0.080	1.053	-1.842	1.493
	$Z_t^{k(\sigma_g^2),inf}$	0.050	1.029	-1.395	1.961	0.144	1.035	-1.261	2.087
	$Z_t^{k(\sigma_g^2),feas}$	-0.233	1.153	-2.270	1.328	-0.217	1.264	-2.291	1.287

Tables 1 to 6 report summary statistics for Levy distribution at times $t = \{10^{-5}, 2.5\times10^{-5}, 5\times10^{-5}, 7.5\times10^{-5}, 10^{-4}, 2.5\times10^{-4}\}$ for the case $c = 1.4\times10^{-5}$ and at times $t = \{10^{-3}, 2.5 \times 10^{-3}, 5 \times 10^{-3}, 7.5 \times 10^{-3}, 10^{-2}, 2.5 \times 10^{-2}\}$ for the case $c = 5 \times 10^{-4}$. Figures 1 and 2 show the histogram and QQ plot of the Z-statistics for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$ with $n = 10,000,000$. The mean and standard deviations of the Z-statistics are respectively close to zero and unity, as expected from the CLTs. We can also see that the quantiles are close to their theoretical value. The results are the best around $t = 10^{-4}$, which is close from the median of the distribution. They start to deteriorate around T_- and T_+, and are not satisfying when $t < T_-$ or $t > T_+$. The CLTs definitely require large n. Overall, the feasible and function Z-statistics are not as good as the infeasible and standard Z-statistics, which is usual. Figure 3 shows the estimated mean of hitting-time variance $\sigma_{t,g}^2$ for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4\times10^{-5}$. This is based on small

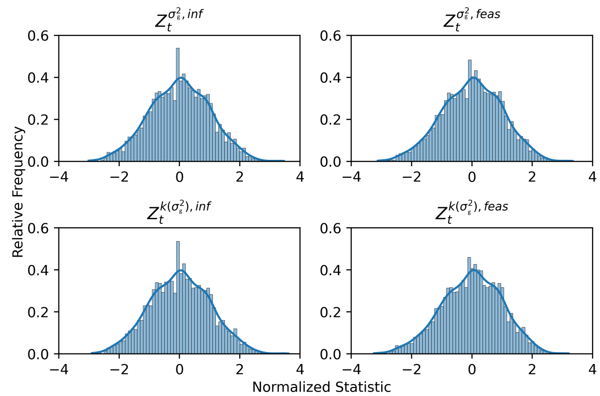


Fig 1: Histogram of the Z-statistics for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$. This is based on $n = 10,000,000$ and $N = 1,000$ iterations.

$n = 1,000$ (left panel) and intermediary $n = 50,000$ (right panel). As expected, the hitting-time variance estimator does not perform as well as in the larger n case.

7. Empirical application

In this section, we conduct an empirical application to financial data. The dataset contains information about transactions of a stock price and the transaction times. More specifically, we restrict to the transactions that correspond to a price change. By duration, we mean the time increment between two consecutive transactions in the dataset. We consider that the next transaction occurs whenever the absolute value of the efficient price increment starting at the previous transaction time hits a constant boundary. The model is based on the FHT introduced in Definition (1). We rely on durations of the stock Apple Inc. (AAPL), which is traded daily from 9:30 AM to 4:00 PM on the Nasdaq stock exchange during the year 2017. The transaction times are concatenated across all the days during the year. This yields $n + 1 = 36,247,426$ transactions.

We first introduce the efficient price. We define the efficient stock price at time t as $\overline{X}_t$. Here, we have $t \in [0, T]$, where the interval $[0, T]$ represents one day of trading from 9:30 AM to 4:00 PM. We assume that the efficient price $\overline{X}_t$ is a local martingale defined as

$$\overline{X}_t = \int_0^t \overline{\sigma}_s dW_s \text{ for any } t \geq 0,$$

Here, the random function $\overline{\sigma}_t$ denotes the volatility. This definition for an efficient price is consistent with a fundamental principle in financial economics, i.e. there is no free lunch with vanishing risk. However, this is a very simplistic

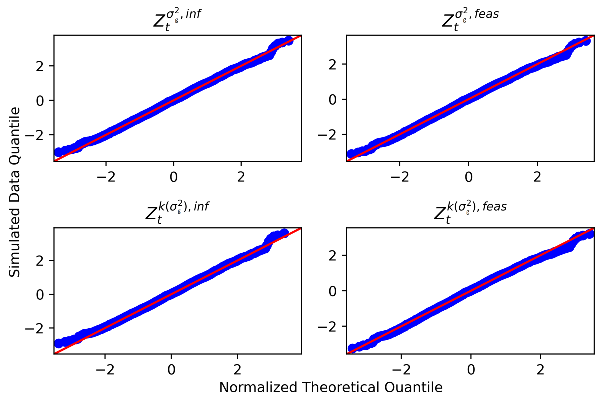


Fig 2: QQ plot of the Z-statistics for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$. This is based on $n = 10,000,000$ and $N = 1,000$ iterations.

(a) $n = 1000$

(b) $n = 50,000$

Fig 3: Estimated mean of hitting-time variance $\sigma^2_{t,g}$ for Levy distribution at time $t = 10^{-4}$ for the case $c = 1.4 \times 10^{-5}$. This is based on $N = 1,000$ iterations, small n (left panel) and intermediary n (right panel).

model of an efficient price for three main reasons. First, the volatility can be random but is very restricted, which prevents well known empirical effects such as the clustering of volatility and the leverage effect. Second, there is no drift term. Moreover, there is no jumps in the efficient price.

We assume that the efficient price process X_t is observed at the random time points t_i for any index $i = 0, \dots, n$. By observations, we mean that there are transactions recorded in the dataset at the random times t_i. We consider that the next transaction occurs whenever the absolute value of the efficient price increment starting at the previous transaction time hits a constant boundary. This can be rewritten recursively as $t_0 = 0$ and $t_{i+1} = \mathrm{T}_g^{|\overline{X} - \overline{X}_{t_i}|}$ for any index $i = 0, \dots, n-1$. Here, we assume for simplicity that the starting time $t_0 = 0$ is a transaction time. We consider that the boundary is equal to the tick size. The tick size is defined as the minimum price increment and is equal to 0.01

USD for AAPL stock. Thus, we fix the boundary to $g = 0.01$. It is natural to choose the tick size as this is the minimum increment or decrement change of the efficient price X_t. This is a simple model of endogenous transaction times as they depend on the efficient price process X_t.

We now rewrite the efficient price $\overline{X}_t$ relying on the stochastic processes $X_t^{(i)} = \int_0^t \sigma_{s,g} dW_s^{(i)}$ for any index $i = 1, \cdots, n$. We also reexpress the transaction times as the FHT introduced in Definition (1). The idea is to use one stochastic process to generate the next transaction time. Thus, we will obtain $n + 1$ transaction times including the first transaction time which does not require any stochastic process $X_t^{(i)}$. More specifically, this can be rewritten recursively as $\overline{X}_t = \overline{X}_{t_i} + X_{t-t_i}^{(i)}$ with $t \in [t_i, t_{i+1})$ for any index $i = 0, \ldots, n-1$. Then, we also have that the transaction times are equal to $t_i = \sum_{k=1}^{i} \mathrm{T}_g^{|X^{(k)}|}$ for any index $i = 0, \ldots, n-1$. Finally, we have that the volatility $\overline{\sigma}_t$ can be reexpressed as a random transformation of the volatility $\sigma_{t,g}$. Namely, we have that $\overline{\sigma}_t = \sigma_{t-t_i,g}$ with $t \in [t_i, t_{i+1})$ for any index $i = 0, \ldots, n-1$. Thus, the volatility jumps at each transaction time provided

$$\sigma_{\mathrm{T}_g^{|X^{(k)}|},g} \neq \sigma_{0,g}.$$

Since for any $i = 1, \cdots, n$ the standard Brownian motions $W^{(i)}$ which are independent and $\mathbf{F}$-adapted, the efficient price $\overline{X}_t$ is also $\mathbf{F}$-adapted by definition.

For our application, we consider hitting-time of the absolute value of the stochastic processes $X_t^{(i)}$. As the methodology considered in this paper is limited to the case of the stochastic processes $X_t^{(i)}$, we cannot use directly the results of this paper. However, the problem of the absolute value of the stochastic processes $X_t^{(i)}$ hitting a boundary can be rewritten as a problem of the stochastic processes $X_t^{(i)}$ hitting a symmetric boundary. Such case is covered by Section 3.2 in Potiron (2023). Thus, the methodology of this paper can be extended easily to the problem of the absolute value of the stochastic processes $X_t^{(i)}$ hitting a boundary. Moreover, we can extend the methodology to the case of random volatility $\sigma_{t,g}$. Indeed, the adaptation would follow from Section 3.3 in Potiron (2023), which considers the random case. Unfortunately, the case of a non zero drift is hard and above the scope of this paper.

Table 7
Summary statistics of the durations in seconds for Apple in 2017. The columns entitled "q%" correspond to the q%-quantile.

	n	Mean	Std	Min	Max	
	36,247,425	1.62×10^{-1}	5.41×10^{-1}	1.00×10^{-6}	7.09×10^{2}	
1%	5%	25%	50%	75%	95%	99%
1.00×10^{-6}	3.00×10^{-6}	1.50×10^{-5}	6.33×10^{-4}	8.47×10^{-2}	9.39×10^{-1}	2.14

Figure 4 shows the histogram of the durations, while Table 7 reports the summary statistics. The histogram of durations has a first pick around one

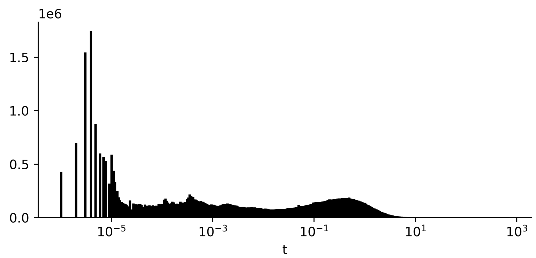


Fig 4: Histogram of the durations in seconds for Apple in 2017

microsecond, a second pick around one millisecond, a third pick around one second and is right-skewed with fat tail. Thus, we believe that a nonparametric model is the most suitable to describe the data. In particular, the data do not seem to follow a Levy distribution.

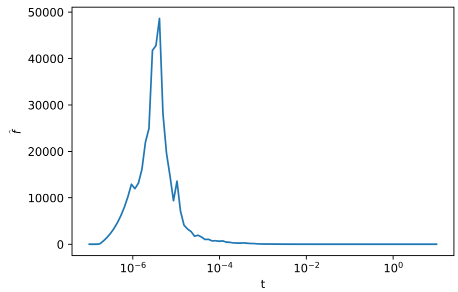
We estimate the pdf and cdf of the durations between $t = 1.00 \times 10^{-6}$ and $t = 7.09 \times 10^{2}$ with $S = 5,000,000$ partitions. We choose α_n from Equation (48). We set $T_{-} = 3.00 \times 10^{-6}$ and $T_{+} = 9.39 \times 10^{-1}$, which covers exactly 90% of the distribution. We also set the number of intervals such that $M_n = \left(\frac{n}{\alpha_n}\right)^{\frac{1}{4}}$. Figure 5 and Table 8 show the estimated pdf and cdf of the FHT.

Figure 6 and Table 9 show the estimated squared volatility. We find that the squared volatility is periodic and follows an inverse J-shape pattern. Such behavior of the squared volatility is empirically well documented at the scale of a day. However, our results are at the scale of a duration, which on average corresponds to 0.15 seconds. This can be explained by the endogeneity of the transaction times.

Since the squared volatility is bounded, it satisfies Assumption 1. Figure 7 and Table 10 show the asymptotic standard deviation as a function of time, i.e.

$$\left(\sqrt{\frac{n}{\alpha_n}\frac{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))^2}{\widehat{f}_n(t)}}\right)^{-1}.$$

We also fit one parametric model, i.e. the Levy distribution. Although we have seen that the data do not follow a Levy distribution, we believe the Levy distribution is moderately suitable since it has right skewed pdf and fat tails. We consider maximum likelihood estimation on the concatenated data in 2017. We find that the estimated scale parameter value of the Levy pdf is equal to $c = 1.40 \times 10^{-5}$, which is used in our numerical simulation.



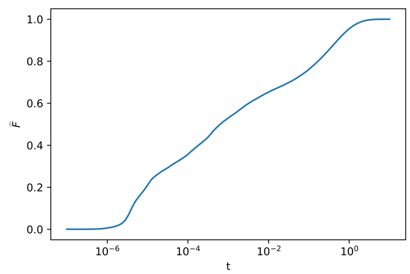


Fig 5: Estimated pdf and cdf of the durations in seconds for Apple in 2017

8. Conclusion

In this paper, we have considered estimation of hitting-time variance. The nonparametric estimation was based on delta-sequences. We have characterized feasible statistics induced by central limit theory for the estimation procedure. An empirical application to financial data documents that the volatility is periodic at the duration scale. This can be explained by the endogeneity of transaction times.

The code is available online at

$https://github.com/Julian-Kota-Kikuchi/volatility\_estimation\_replication$

TABLE 8
Estimated pdf and cdf of the durations in seconds for Apple in 2017

t	$\widehat{f}_n$	$\widehat{F}_n$
1.00×10^{-7}	0.00	2.76×10^{-8}
1.00×10^{-6}	1.40×10^{4}	5.83×10^{-3}
1.00×10^{-5}	1.94×10^{4}	2.08×10^{-1}
1.00×10^{-4}	7.92×10^{2}	3.56×10^{-1}
1.00×10^{-3}	8.27×10	5.30×10^{-1}
1.00×10^{-2}	5.18	6.53×10^{-1}
1.00×10^{-1}	2.89	7.61×10^{-1}
1.00	2.33×10^{-1}	9.55×10^{-1}
1.00×10	0.00	1.00

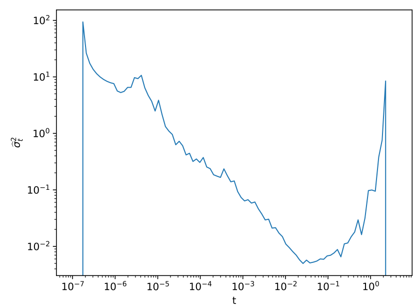


Fig 6: Estimated squared volatility for the boundary value $g = 0.01$ in seconds for Apple in 2017.

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Supplementary Material

Supplement: Parametric case
The supplement specifies the hitting-time variance for some standard parametric distribution

Supplement: Proofs
The supplement provides the detailed proofs of the results from Section 4.

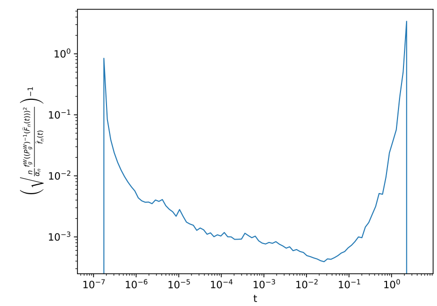

Fig 7: Estimated asymptotic standard deviation of the squared volatility for the boundary value $g = 0.01$ in seconds for Apple in 2017 in seconds for Apple in 2017

TABLE 9
Estimated asymptotic standard deviation of the squared volatility for the boundary value $g = 0.01$ in seconds for Apple in 2017

t	
1.00×10^{-7}	0.00
1.00×10^{-6}	7.52
1.00×10^{-5}	5.38
1.00×10^{-4}	3.86×10^{-1}
1.00×10^{-3}	1.01×10^{-1}
1.00×10^{-2}	1.58×10^{-2}
1.00×10^{-1}	2.69×10^{-2}
1.00	3.15×10^{-1}
1.00×10	0.00

They rely on theory developed in Watson and Leadbetter (1964b) Doob (1949), Malmquist (1954), Loeve (1977) and Wang and Pötzelberger (1997).

Table 10
Estimated asymptotic standard deviation of the squared volatility for the boundary value $g = 0.01$ in seconds for Apple in 2017

t	
1.00×10^{-7}	0.00
1.00×10^{-6}	4.08×10^{-1}
1.00×10^{-5}	3.45×10^{-1}
1.00×10^{-4}	9.25×10^{-2}
1.00×10^{-3}	4.75×10^{-2}
1.00×10^{-2}	1.87×10^{-2}
1.00×10^{-1}	2.44
1.00	8.35×10^{-2}
1.00×10	0.00

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SUPPLEMENTARY MATERIAL: This is the supplementary material of ”Nonparametric estimation of hitting-time variance” by Julian Kota Kikuchi, Chang Yuan Li and Yoann Potiron submitted to the Annals of the Institute of Statistical Mathematics. This supplementary material specifies the hitting-time variance for some standard parametric distribution in Section 9 and provides the detailed proofs of the results from Section 4 in Section 10.

9. Parametric case

This supplement specifies the hitting-time variance for some standard parametric pdfs such as Levy pdfs, inverse gamma pdfs, inverse Gaussian pdfs, gamma pdfs and pdfs for the FHT of a standard Brownian motion to a linear boundary.

We first introduce some notation. We define the error function and its inverse as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt \text{ for any } z \in \mathbb{R}, \tag{49}$$

$$\mathrm{erf}(\mathrm{erfinv}(z)) = z \text{ for any } z \in (-1,1). \tag{50}$$

If we introduce the notation $h(t) = \mathrm{erfinv}\left\{1 - F(t)\right\}$ and substitute

$$f_g^W((P_g^W)^{-1}(F(t)))$$

in Equation (4) with Equation (68) from Lemma 2, we can reexpress the hitting-time variance as a closed-form expression, i.e.

$$\sigma_{t,g}^2 = \frac{f(t)}{\frac{2}{g^2\sqrt{\pi}}h(t)^3 e^{-h(t)^2}}\mathbf{1}_{\{0<F(t)<1\}} \quad \text{for any } t \geq 0. \tag{51}$$

9.1. Levy pdf

The Levy pdf and cdf satisfy $f(0) = 0$, $F(0) = 0$,

$$f(t) = \sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2t}}}{t^{\frac{3}{2}}} \text{ for any } t > 0, \tag{52}$$

$$F(t) = 1 - \mathrm{erf}\left(\sqrt{\frac{c}{2t}}\right) \text{ for any } t > 0. \tag{53}$$

Here, $c > 0$ is the scale parameter. It is right skewed, has an infinite mean and a fat tail with an asymptotic polynomial decay of order $t^{-\frac{3}{2}}$. By Equations (65) and (66) from Lemma 2, it corresponds to the pdf of the FHT by a standard Brownian motion to the boundary $g = \sqrt{c}$. We can reexpress the hitting-time

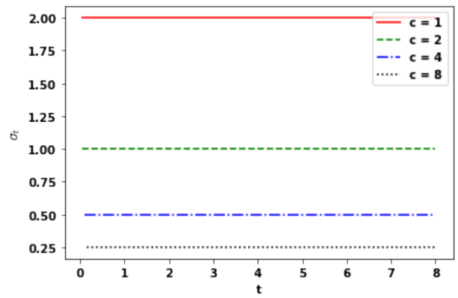

Fig 8: Square root of hitting-time variance related to Levy distribution for several scale parameters

variance for any $t \geq 0$ as

$$\begin{aligned}
\sigma_{t,g}^2 &= \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\mathbf{1}_{\{0<F(t)<1\}} \\
&= \frac{f(t)}{f_g^W((P_g^W)^{-1}(1-\text{erf}(\sqrt{\frac{c}{2t}})))}\mathbf{1}_{\{t>0\}} \\
&= \frac{f(t)}{f_g^W\left(\frac{g^2}{2(\text{erfinv}(\text{erf}(\frac{\sqrt{c}}{2t})))^2}\right)}\mathbf{1}_{\{t>0\}} \\
&= \frac{f(t)}{f_g^W(\frac{g^2}{c}t)}\mathbf{1}_{\{t>0\}} \\
&= \frac{g^2}{c}\mathbf{1}_{\{t>0\}}, \qquad (54)
\end{aligned}$$

where the first equality corresponds to Equation (4), the second equality is due to Equation (53), the third equality comes from Equation (67) in Lemma 2, the fourth equality is obtained with algebraic manipulation, we use Lemma 2 along with Equation (52) in the fifth equality. The hitting-time variance is constant since the square of the boundary and the scale parameter play a symmetric role. Figure 8 shows the square root of hitting-time variance related for several scale parameters.

9.2. *Inverse gamma pdf*

The inverse gamma pdf and cdf satisfy $f(0) = 0$, $F(0) = 0$,

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{e^{-\frac{\beta}{t}}}{t^{\alpha+1}} \text{ for any } t > 0, \qquad (55)$$

$$F(t) = \frac{\Gamma\left(\alpha, \frac{\beta}{t}\right)}{\Gamma(\alpha)} \text{ for any } t > 0. \qquad (56)$$

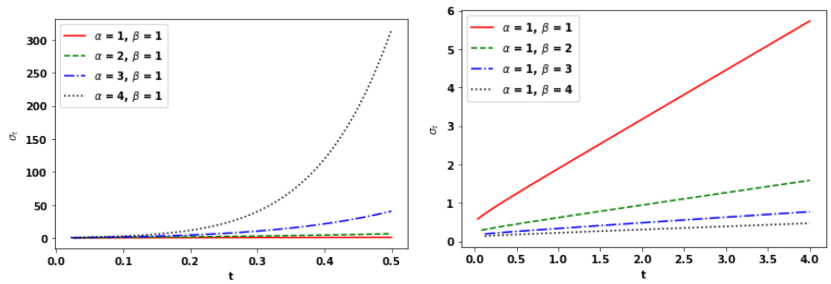

Fig 9: Square root of hitting-time variance related to the the inverse gamma distribution for several shape and scale parameters

Here, $\alpha > 0$ is the shape parameter, $\beta > 0$ is the scale parameter, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function, and $\Gamma(\alpha, x) = \int_x^\infty x^{\alpha-1}e^{-x}dx$ is the upper incomplete gamma function. It is right skewed and has a fat tail with an asymptotic polynomial decay of order $t^{-1-\alpha}$. When $0 < \alpha < 1/2$, the polynomial decay is slower than the decay of the Levy pdf. When $\alpha = 1/2$, the polynomial decay is the same and in fact both pdfs are equal when $\beta = c/2$. When $0 < \alpha < 1/2$, the polynomial decay is faster than the decay of the Levy pdf. By Equation (51) and Equations (55) and (56), we can reexpress the hitting-time variance for any $t \geq 0$ as

$$\sigma_{t,g}^2 = \frac{\beta^\alpha g^2 \sqrt{\pi}}{2\Gamma(\alpha)} \frac{e^{h(t)^2 - \frac{\beta}{t}}}{t^{\alpha+1} h(t)^3} \mathbf{1}_{\{t>0\}}.$$

Figure 9 shows the square root of hitting-time variance related for several shape and scale parameters.

9.3. Inverse Gaussian pdf

The inverse Gaussian pdf and cdf satisfy $f(0) = 0$, $F(0) = 0$,

$$f(t) = \sqrt{\frac{\lambda}{2\pi}} \frac{e^{-\frac{\lambda(t-\mu)^2}{2\mu^2 t}}}{t^{\frac{3}{2}}} \text{ for any } t > 0, \quad (57)$$

$$F(t) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right) + e^{\frac{2\lambda}{\mu}} \Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right) \text{ for any } t > 0 \quad (58)$$

Here, $\lambda > 0$ is the shape parameter, $\mu > 0$ is the mean parameter and Φ is defined as the cdf of the standard Gaussian distribution, i.e.

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \text{ for any } z \in \mathbb{R}. \quad (59)$$

It has an asymptotic exponential decay. It corresponds to the pdf of the FHT of a Brownian motion with drift $\nu = g/\mu$ and variance $\sigma^2 = g^2/\lambda$ to the boundary

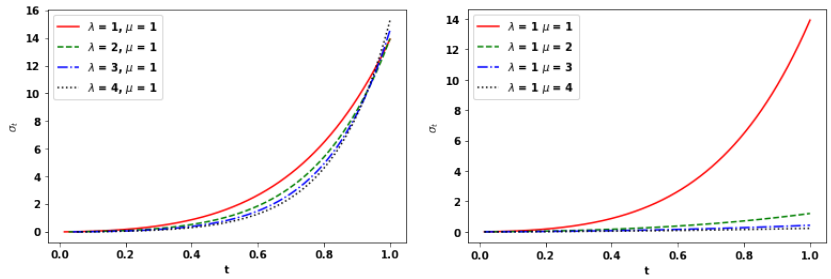

Fig 10: Square root of hitting-time variance related to the inverse Gaussian distribution for several shape and mean parameters

g. By Equation (51) and Equations (57)-(58), we can reexpress the hitting-time variance for any $t \geq 0$ as

$$\sigma_{t,g}^2 = \frac{g^2\sqrt{\lambda}}{2\sqrt{2}} \frac{e^{h(t)^2 - \frac{\lambda(t-\mu)^2}{2\mu^2 t}}}{t^{\frac{3}{2}} h(t)^3} \mathbf{1}_{\{t>0\}}.$$

Figure 10 shows the square root of hitting-time variance for several shape and mean parameters.

9.4. Gamma pdf

The gamma pdf and cdf satisfy

$$f(t) = \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \text{ for any } t \geq 0, \tag{60}$$

$$F(t) = \frac{\int_0^{\beta t} x^{\alpha-1} e^{-x} dx}{\Gamma(\alpha)} \text{ for any } t \geq 0. \tag{61}$$

Here, $\alpha > 0$ is the shape parameter and $\beta > 0$ is the rate parameter. When $\alpha = 1$, it becomes exponential whose pdf and cdf satisfy

$$f(t) = \beta e^{-\beta t} \text{ for any } t \geq 0,$$
$$F(t) = 1 - e^{-\beta t} \text{ for any } t \geq 0.$$

The gamma pdf has an asymptotic exponential decay. By Equation (51) and Equation (60)-(61), we can reexpress the hitting-time variance for any $t \geq 0$ as

$$\sigma_{t,g}^2 = \frac{\beta^\alpha g^2 \sqrt{\pi}}{2\Gamma(\alpha)} \frac{t^{\alpha-1} e^{h(t)^2 - \beta t}}{h(t)^3}. \tag{62}$$

When $\alpha = 1$, the closed-form solution can be specified as

$$\sigma_{t,g}^2 = \frac{\beta g^2 \sqrt{\pi}}{2} \frac{e^{\text{erfinv}(e^{-\beta t})^2 - \beta t}}{\text{erfinv}(e^{-\beta t})^3}.$$

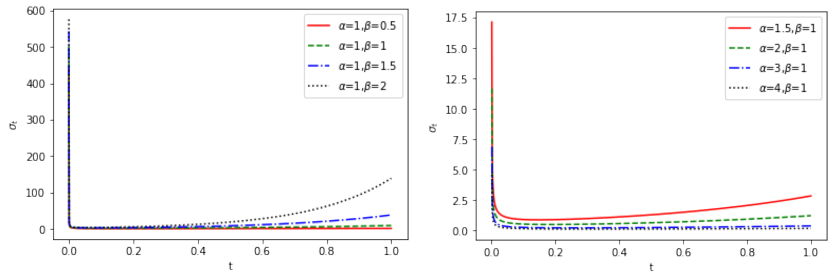

Fig 11: Square root of hitting-time variance related to the exponential distribution (left panel) and the gamma distribution (right panel) for several shape and rate parameters

Figure 11 shows the square root of hitting-time variance for several shape and rate parameters.

9.5. Linear boundary pdf

The next lemma gives the pdf and cdf of the FHT by a standard Brownian motion to the linear boundary $g(t) = at + b$, where $a \in \mathbb{R}$ is the slope and $b > 0$ is the intercept, which are known results from Malmquist (1954) (Theorem 1, p. 526).

Lemma 1. *We obtain* $f(0) = 0$, $F(0) = 0$,

$$f(t) = \frac{be^{-\frac{(at+b)^2}{2t}}}{\sqrt{2\pi t^3}} \text{ for any } t > 0. \tag{63}$$

$$F(t) = 1 - \Phi\left(\frac{at+b}{\sqrt{t}}\right) + e^{-2ab}\Phi\left(\frac{at-b}{\sqrt{t}}\right) \text{ for any } t > 0. \tag{64}$$

When $a = 0$, the boundary becomes constant and we obtain a Levy distribution with parameter b^2 as in Lemma 2 and Section 9.1. When $a \neq 0$, the pdf has an exponential decay. By Equations (51),(63) and (64), we can reexpress the hitting-time variance for any $t \geq 0$ as

$$\sigma_{t,g}^2 = \frac{g^2 be^{-\frac{(at+b)^2}{2t}}}{2\sqrt{2t^3}} \frac{e^{\mathrm{erfinv}(h(t))^2}}{\mathrm{erfinv}\,(h(t))^3}.$$

Figures 12 and 13 show the square root of hitting-time variance for several slope and intercept parameters.

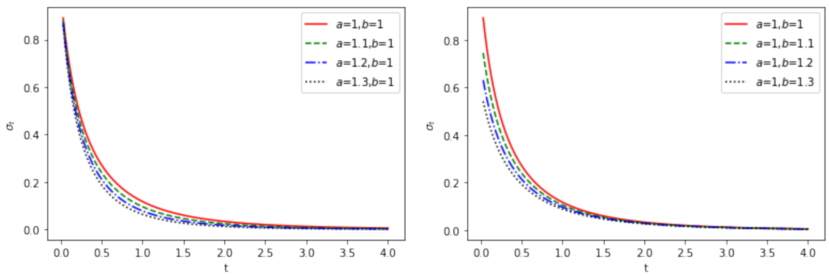

Fig 12: Square root of hitting-time variance related to the hitting-time of a linear upper boundary of the form $at+b$ by a Brownian motion for several slope and intercept parameters

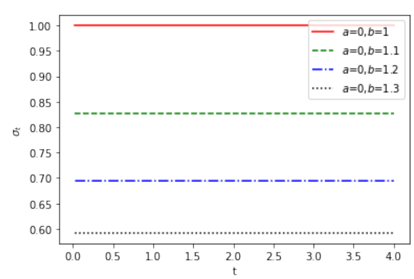

Fig 13: Square root of hitting-time variance related to the hitting-time of a constant upper boundary of the form b by a Brownian motion for several intercept parameters

10. Proofs

This supplement provides the detailed proofs of the results from Section 4.

To obtain the form of the variance solution in the IFHT problem, the key result in Potiron (2023) states that the cdf for the FHT of a continuous local martingale is equal to the cdf for the FHT of a standard Brownian motion, which is time-changed by the martingale quadratic variation. Thus, we now consider the case when $Z_t = W_t$ for any $t \in \mathbb{R}^+$ and recall a basic lemma from Potiron (2023). This lemma gives f_g^Z and P_g^Z, i.e. Levy distribution, in Equations (65)-(66) which are known results from Malmquist (1954) (Theorem 1, p. 526). It also shows that there exists an invert of P_g^W which we denote $(P_g^W)^{-1}$ and gives $(P_g^W)^{-1}(t)$ and $f_g^W((P_g^W)^{-1}(t))$ for any $0 \leq t < 1$.

Lemma 2. *We obtain a Levy distribution with $f_g^W(0) = 0$, $P_g^W(0) = 0$,*

$$f_g^W(t) = \frac{g}{\sqrt{2\pi t^3}} e^{\frac{-g^2}{2t}} \text{ for any } t > 0, \tag{65}$$

$$P_g^W(t) = 1 - erf\left(\frac{g}{\sqrt{2t}}\right) \text{ for any } t > 0. \tag{66}$$

Moreover, there exists an invert of P_g^W which we denote $(P_g^W)^{-1} : [0,1) \to \mathbb{R}^+$ and is strictly increasing such that $(P_g^W)^{-1}(0) = 0$,

$$(P_g^W)^{-1}(t) = \frac{g^2}{2\,\mathrm{erfinv}(1-t)^2} \text{ for any } 0 < t < 1. \tag{67}$$

Finally, we have $f_g^W((P_g^W)^{-1}(t)) = 0$,

$$f_g^W((P_g^W)^{-1}(t)) = \frac{2}{g^2\sqrt{\pi}} \mathrm{erfinv}(1-t)^3 e^{-\mathrm{erfinv}(1-t)^2} \text{ for any } 0 < t < 1. \tag{68}$$

Proof of Lemma 2. This corresponds to Lemma 6 in Potiron (2023). □

The next lemma is a direct consequence of the previous lemma.

Lemma 3. *We have*

$$f_g^W \in \mathcal{C}_1(\mathbb{R}^+, \mathbb{R}^+) \text{ and } P_g^W \in \mathcal{C}_1(\mathbb{R}^+, [0,1)), \tag{69}$$

$$(P_g^W)^{-1} \in \mathcal{C}_1([0,1), \mathbb{R}^+). \tag{70}$$

Proof of Lemma 3. This is a direct consequence of Lemma 2. □

We provide the following lemma, which corresponds to the L^2 convergence of $\widehat{F}_n(t)$ to the limit $F(t)$.

Lemma 4. *We assume that Assumption 2 holds. We have that the L^2 convergence of $\widehat{F}_n(t)$ to the limit $F(t)$, i.e. for $t \geq 0$ that*

$$\mathbb{E}[(\widehat{F}_n(t) - F(t))^2] \to 0.$$

Proof of Lemma 4. We can express the L^2-distance as the sum of the square of the bias and the variance, i.e.

$$\mathbb{E}[(\widehat{F}_n(t) - F(t))^2] \quad = \quad (\mathbb{E}[\widehat{F}_n(t)] - F(t))^2 + \mathrm{Var}[\widehat{F}_n(t)].$$

Then, the first term and the second term in the right-hand side of the above equation tend to 0 respectively by Theorem 6 (i) and Theorem 6 (ii) in Watson and Leadbetter (1964b) (p. 112), along with Assumption 2. □

We give another lemma, which establishes the L^2 convergence of $\widehat{f}_n(t)$ to the limit $f(t)$.

Lemma 5. *We assume that Assumption 2 and Assumption 3 Expressions (19)-(20)-(21)-(22) hold. We have L^2 convergence of $\widehat{f}_n(t)$ to the limit $f(t)$, i.e. for any $t \geq 0$ assuming that it is a continuity point of f, that*

$$\mathbb{E}[(\widehat{f}_n(t) - f(t))^2] \to 0.$$

Proof of Lemma 5. As L^2-convergence implies $\mathbb{P}$-convergence, it is sufficient to show the L^2-convergence. We have that

$$\mathbb{E}[(\widehat{f}_n(t) - f(t))^2] \quad = \quad (\mathbb{E}[\widehat{f}_n(t)] - f(t))^2 + \mathrm{Var}[\widehat{f}_n(t)].$$

Then, the squared bias term vanishes asymptotically as an application of Theorem 5 (p. 111) from Watson and Leadbetter (1964b) (p. 112) and the variance term goes to 0 using Theorem 4 (p. 110) from the aforementioned paper, together with Assumption 2, Assumption 3 Expressions (19)-(20)-(21)-(22) and the fact that t is a continuity point of f. □

We now give the proof of Theorem 1.

Proof of Theorem 1. First, we prove the standard CLT (27). We can write for any $t > 0$, assuming that it is a continuity point of f, that

$$\begin{aligned}
\sqrt{\frac{n}{\alpha_n}}(\widehat{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g}) \quad &= \quad \sqrt{\frac{n}{\alpha_n}}\Big(\widehat{\sigma}^{2,n}_{t,g} - \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\
&= \quad \sqrt{\frac{n}{\alpha_n}}\Big(\frac{\widehat{f}_n(t)\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\
&= \quad \sqrt{\frac{n}{\alpha_n}}\Big(\frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(F(t)))} - \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\
&\quad + \sqrt{\frac{n}{\alpha_n}}\Big(\frac{\widehat{f}_n(t)\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\
&= \quad I + II.
\end{aligned}$$

Here, the first equality is due to Equation (4) along with the assumption that $f(t) > 0$ and $t > 0$, the second equality corresponds to Equation (7), and the third equality is obtained with algebraic manipulation. In what follows, we will show that

$$I \overset{\mathcal{D}}{\to} \mathcal{N}\Big(0, \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2}\Big), \tag{71}$$

$$II \overset{\mathbb{P}}{\to} 0. \tag{72}$$

We start with the proof of I. We can decompose it as

$$\begin{aligned} I &= \sqrt{\frac{n}{\alpha_n}}\Big(\frac{\widehat{f}_n(t) - \mathbb{E}[\widehat{f}_n(t)])}{f_g^W((P_g^W)^{-1}(F(t)))} - \frac{f(t) - \mathbb{E}[\widehat{f}_n(t)])}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\ &:= I_A + I_B. \end{aligned}$$

We can see that $I_B \overset{\mathbb{P}}{\to} 0$ by Assumption 3 Expression (23). We turn now to the I_A term. Note that by Theorem 5 (p. 111) in Watson and Leadbetter (1964b) together with Assumption 2, Assumption 3 Expressions (19)-(20)-(21)-(22) and the assumption that t is a continuity point of f, we obtain that

$$\sqrt{\frac{n}{\alpha_n}}(\widehat{f}_n(t) - \mathbb{E}[\widehat{f}_n(t)]) \overset{\mathcal{D}}{\to} \mathcal{N}(0, f(t)). \tag{73}$$

We have thus shown Equation (71).

We now continue with the proof of II. With an algebraic manipulation, we can decompose it as

$$\begin{aligned} II &= \sqrt{\frac{n}{\alpha_n}}\Big(\frac{(\widehat{f}_n(t) - \mathbb{E}[\widehat{f}_n(t)])\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{\widehat{f}_n(t) - \mathbb{E}[\widehat{f}_n(t)]}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\ &\quad + \sqrt{\frac{n}{\alpha_n}}\Big(\frac{(\mathbb{E}[\widehat{f}_n(t)] - f(t))\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{\mathbb{E}[\widehat{f}_n(t)] - f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\ &\quad + \sqrt{\frac{n}{\alpha_n}}\Big(\frac{f(t)\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\ &= II_A + II_B + II_C. \end{aligned}$$

We start with II_A term. First, we note that by Theorem 5 (p. 111) in Watson and Leadbetter (1964b) along with Assumption 2, Assumption 3 Expressions (19),(20),(21) and (22), and the fact that t is a continuity point of f, we obtain

$$\sqrt{\frac{n}{\alpha_n}}(\widehat{f}_n(t) - \mathbb{E}[\widehat{f}_n(t)]) = O_{\mathbb{P}}(1). \tag{74}$$

As a consequence of Equation (74), proving that $II_A \overset{\mathbb{P}}{\to} 0$ amounts to showing that

$$\frac{\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} \overset{\mathbb{P}}{\to} \frac{1}{f_g^W((P_g^W)^{-1}(F(t)))}. \tag{75}$$

Equation (75) holds as an application of Lemma 4 along with the continuous mapping theorem and the fact that f_g^W and $(P_g^W)^{-1}$ are continuous functions by Lemma 2. This is also due to the assumption that $f(t) > 0$ and $t > 0$.

We turn to II_B term. First, we can see that by Lemma 4 along with the assumption that $f(t) > 0$ and $t > 0$ we have

$$\frac{\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} = O_{\mathbb{P}}(1). \tag{76}$$

From Equation (76), the proof of $II_B \xrightarrow{\mathbb{P}} 0$ amounts to showing that the bias vanishes asymptotically, i.e. that

$$\sqrt{\frac{n}{\alpha_n}}(\mathbb{E}[\widehat{f}_n(t)] - f(t)) = o_{\mathbb{P}}(1), \tag{77}$$

which corresponds exactly to Assumption 3 Expression (23).

For II_C term, we define the function $u : (0,1) \to \mathbb{R}_*^+$ as

$$u(s) = \frac{1}{f_g^W((P_g^W)^{-1}(s))}. \tag{78}$$

First, we note that $u \in \mathcal{C}_1((0,1), \mathbb{R}_*^+)$ since $f_g^W \in \mathcal{C}_1(\mathbb{R}^+, \mathbb{R}^+)$ and $(P_g^W)^{-1} \in \mathcal{C}_1([0,1), \mathbb{R}^+)$ by Expressions (69)-(70) from Lemma 3. By the use of a Taylor expansion, we can deduce that

$$u(\widetilde{F}_n(t)) - u(F(t)) = u'(s_n)(\widetilde{F}_n(t) - F(t)). \tag{79}$$

Here, we introduce the truncated cdf estimator

$$\widetilde{F}_n(t) = \min\big(1 - (1 - F(t))/2, \max(\widehat{F}_n(t), F(t)/2)\big)$$

and s_n is between $\widetilde{F}_n(t)$ and $F(t)$. We have

$$\begin{aligned} II_C &= \sqrt{\frac{n}{\alpha_n}}\Big(\frac{f(t)\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))} - \frac{f(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big) \\ &= \sqrt{\frac{n}{\alpha_n}} f(t)\big(u(\widetilde{F}_n(t)) - u(F(t))\big) + o_{\mathbb{P}}(1) \\ &= \sqrt{\frac{n}{\alpha_n}} f(t) u'(s_n)(\widehat{F}_n(t) - F(t)) + o_{\mathbb{P}}(1), \end{aligned}$$

where the first equality is the definition of II_C, the second equality is due to Equation (78) with Lemma 4 and the assumption that $f(t) > 0$ where $t > 0$, and the third equality is obtained from Equation (79) with Lemma 4 the assumption that $f(t) > 0$ where $t > 0$. Then, we have that $\mid u'(s_n) \mid \leq C$ as s_n belongs to a compact space $[s_-, s_+]$ such that $0 < s_- < s_+ < 1$ by the assumption $f(t) > 0$ where $t > 0$. Thus, we can deduce that

$$\mid II_C \mid \leq C\sqrt{\frac{n}{\alpha_n}} \mid \widehat{F}_n(t) - F(t) \mid. \tag{80}$$

By an algebraic manipulation, we can decompose $\widehat{F}_n(t) - F(t)$ as

$$\widehat{F}_n(t) - F(t) = (\widehat{F}_n(t) - \mathbb{E}[\widehat{F}_n(t)]) + (\mathbb{E}[\widehat{F}_n(t)] - F(t)). \tag{81}$$

As a consequence of Equation (80) and Equation (81), along with the triangular inequality, we deduce that

$$| II_C | \leq C\sqrt{\frac{n}{\alpha_n}}(| \widehat{F}_n(t) - \mathbb{E}[\widehat{F}_n(t)] | + | \mathbb{E}[\widehat{F}_n(t)] - F(t) |). \tag{82}$$

First, we note that by Theorem 6 (iii) (p. 112) in Watson and Leadbetter (1964b) together with Assumption 2, we obtain

$$\sqrt{n}(\widehat{F}_n(t) - \mathbb{E}[\widehat{F}_n(t)]) = O_{\mathbb{P}}(1). \tag{83}$$

Then, from Equation (83) along with the fact that $\alpha_n \to \infty$, i.e. Expression (26), we can deduce that the first term in the right-hand side of Equation (82) tends to 0 in probability. The second term in the right-hand side of Equation (82) is a bias term and vanishes asymptotically by Assumption 3 Expression (24). Thus we have shown Equation (72). We have thus shown the standard CLT (27).

Now, we prove the feasible normalized CLT (28). First, as an application of the continuous mapping theorem along with Lemma 4, Lemma 5 and the fact that f_g^W and $(P_g^W)^{-1}$ are continuous functions by Expressions (69)-(70) from Lemma 3, we obtain that

$$\sqrt{\frac{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))^2}{\widehat{f}_n(t)}} \xrightarrow{\mathbb{P}} \sqrt{\frac{f_g^W((P_g^W)^{-1}(F(t)))^2}{f(t)}}. \tag{84}$$

Now, an application of Equation (84), the standard CLT (27) and Slutsky's lemma yields

$$\sqrt{\frac{n}{\alpha_n}\frac{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))^2}{\widehat{f}_n(t)}}(\widehat{\sigma}_{t,g}^{2,n} - \sigma_{t,g}^2) \tag{85}$$
$$\xrightarrow{\mathcal{D}} \sqrt{\frac{f_g^W((P_g^W)^{-1}(F(t)))^2}{f(t)}}\mathcal{N}(0, \frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}).$$

Then, as $\sqrt{a}\mathcal{N}(0, b) = \mathcal{N}(0, ab)$ for any $a > 0$ and $b > 0$, we can deduce that the term in the right-hand side of Expression (85) is equal to $\mathcal{N}(0, 1)$. ☐

We now give the proof of Theorem 2.

Proof of Theorem 2. We first show the standard CLT (45). First, we define the hitting-time variance estimator truncated on the interval $[\sigma_{t,g}^2 - \eta, \sigma_{t,g}^2 + \eta]$ as

$$\overline{\sigma}_{t,g}^{2,n} = \min\left(\sigma_{t,g}^2 + \eta, \max(\widehat{\sigma}_{t,g}^{2,n}, \sigma_{t,g}^2 - \eta)\right)$$

Then, we have

$$\begin{aligned}\sqrt{\frac{n}{\alpha_n}}(\widehat{k(\sigma^2_{t,g})}^n - k(\sigma^2_{t,g})) &= \sqrt{\frac{n}{\alpha_n}}(k(\widehat{\sigma}^{2,n}_{t,g})\mathbf{1}_{\{\widehat{\sigma}^{2,n}_{t,g}>0\}} - k(\sigma^2_{t,g})) \\ &= \sqrt{\frac{n}{\alpha_n}}(k(\overline{\sigma}^{2,n}_{t,g}) - k(\sigma^2_{t,g})) + o_{\mathbb{P}}(1),\end{aligned}$$

where the first equality comes from Equation (8) and the second equality comes from the assumption that $f(t) > 0$ with $t > 0$ and that f is continuous at time t. Then, by a Taylor expansion along with Assumption 4 Expression (29) we obtain that

$$\begin{aligned}\sqrt{\frac{n}{\alpha_n}}(k(\overline{\sigma}^{2,n}_{t,g}) - k(\sigma^2_{t,g})) &= \sqrt{\frac{n}{\alpha_n}}k'(\sigma^2_{t,g})(\overline{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g}) \\ &\quad + \sqrt{\frac{n}{\alpha_n}}\frac{k''(\widetilde{\sigma}^2_n)}{2}(\overline{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g})^2 \\ &:= I + II,\end{aligned}$$

where $\widetilde{\sigma}^2_n$ is between $\overline{\sigma}^{2,n}_{t,g}$ and $\sigma^2_{t,g}$. By an application of Theorem 1, Assumptions 1, 2, 3, the assumption that $f(t) > 0$ with $t > 0$ and that f is continuous at time t, we obtain the convergence

$$I \xrightarrow{\mathcal{D}} g'(\sigma^2_{t,g})\mathcal{N}\Big(0, \frac{f(t)}{f^W_g((P^W_g)^{-1}(F(t)))^2}\Big).$$

Given the fact that $\sqrt{a}\mathcal{N}(0,b) = \mathcal{N}(0,ab)$ for any $a > 0$ and $b > 0$, we can deduce that

$$I \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{k'(\sigma^2_{t,g})^2 f(t)}{f^W_g((P^W_g)^{-1}(F(t)))^2}\Big).$$

As for II, we can calculate that

$$\begin{aligned}II &= \sqrt{\frac{n}{\alpha_n}}\frac{k''(\widetilde{\sigma}^2_n)}{2}(\overline{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g})^2 \\ &\leq C\sqrt{\frac{n}{\alpha_n}}(\overline{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g})^2 + o_{\mathbb{P}}(1) \\ &= O_{\mathbb{P}}(\overline{\sigma}^{2,n}_{t,g} - \sigma^2_{t,g}) \\ &= o_{\mathbb{P}}(1),\end{aligned}$$

where the first equality comes from the definition of II, the inequality is due to Assumption 4 Expression (30) and that $\widetilde{\sigma}^2_n$ belongs to the compact interval $[\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta]$, the second and third equalities are consequence of Theorem 1 by Assumptions 1-2-3 along with the assumption that $f(t) > 0$ with $t > 0$ and that f is continuous at time t. Thus we have proven the standard CLT (45).

Now, we prove the feasible normalized CLT (46). First, by Assumption 4 Expression (30) we can deduce that $k' \in \mathcal{C}_2([\sigma^2_{t,g} - \eta, \sigma^2_{t,g} + \eta], \mathbb{R})$ and thus apply the standard CLT (45) to the function k' and obtain

$$k'(\overline{\sigma}^{2,n}_{t,g}) \xrightarrow{\mathbb{P}} k'(\sigma^2_{t,g}).$$

Then, the proof follows with extending the arguments from the normalized feasible CLT in the proof of Theorem 1. □

Before we turn to the proof of Theorem 3, we introduce several lemmas. The next lemma extends Lemma 3 (p. 104) in Watson and Leadbetter (1964b) to a statement uniform on $[T_-, T_+]$.

Lemma 6. *We assume that Assumption 2 and Assumption 5 Expressions (34),(35) and (36) hold . Then for any $t \in [T_-, T_+]$ we have $s \mapsto f(s)\delta_n(s-t) \in L_1(\mathbb{R}^+)$ and*

$$\sup_{T_- \leq t \leq T_+} \Big| \int_{\mathbb{R}^+} \delta_n(s-t) f(s) ds - f(t) \Big| \to 0. \tag{86}$$

Proof of Lemma 6. First, we show that for any $t \in [T_-, T_+]$ we have $s \mapsto f(s)\delta_n(s-t) \in L_1(\mathbb{R}^+)$. Let $t \in [T_-, T_+]$. Given that f is continuous and positive in t by Assumption 5 Expressions (35) and (36), that $f \in L_1(\mathbb{R}^+)$ since it is a density, δ_n is a positive delta-sequence, then $s \mapsto f(s)\delta_n(s-t) \in L_1(\mathbb{R}^+)$ as an application of Lemma 3 (p. 104) in Watson and Leadbetter (1964b).
Then, we prove Expression (86). By Assumption 2 Equation (15), we can reexpress the expression as

$$\begin{aligned} &\sup_{T_- \leq t \leq T_+} \Big| \int_{\mathbb{R}^+} \delta_n(s-t) f(s) ds - f(t) \Big| \\ = &\sup_{T_- \leq t \leq T_+} \Big| \int_{\mathbb{R}^+} \delta_n(s-t)(f(s) - f(t)) ds \Big|. \end{aligned}$$

Note that f is continuous on $[T_-, T_+]$ by Assumption 5 Expression (35), which is a compact space of $\mathbb{R}^+$ by Assumption 5 Expression (34). Thus f is uniformly continuous on $[T_-, T_+]$. Then, for any arbitrary small $\epsilon > 0$, $\lambda > 0$ may be chosen so that

$$\sup_{T_- \leq t \leq T_+, |s-t| \leq \lambda} | f(s) - f(t) | < \epsilon. \tag{87}$$

If we define A as $A = \sup_{T_- \leq t \leq T_+} \left| \int_{\mathbb{R}^+} \delta_n(s-t)(f(s)-f(t))ds \right|$, we obtain that

$$\begin{aligned} A &= \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|\leq\lambda} \delta_n(s-t)(f(s)-f(t))ds \\ &\quad + \int_{|s-t|>\lambda} \delta_n(s-t)(f(s)-f(t))ds \Big| \\ &\leq \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|\leq\lambda} \delta_n(s-t)(f(s)-f(t))ds \Big| \\ &\quad + \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t)(f(s)-f(t))ds \Big| \\ &= I + II, \end{aligned}$$

where we use algebraic manipulation in the equality, and the triangular inequality in the inequality. On the one hand, we have that

$$\begin{aligned} I &\leq \sup_{T_- \leq t \leq T_+} \int_{|s-t|\leq\lambda} \delta_n(s-t)ds \sup_{s\in\mathbb{R}^+, |s-t|\leq\lambda} \mid f(s) - f(t) \mid \\ &\leq \sup_{T_- \leq t \leq T_+, |s-t|\leq\lambda} \mid f(s) - f(t) \mid \\ &\leq \epsilon, \end{aligned}$$

where we use algebraic manipulation with sup domination in the first inequality, Assumption 2 Equation (15) in the second inequality, and Equation (87) in the last inequality. Then, if we choose $\epsilon \to 0$, we obtain that $I \to 0$.
On the other hand, we have that

$$\begin{aligned} II &= \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t)f(s)ds - \int_{|s-t|>\lambda} \delta_n(s-t)f(t)ds \Big| \\ &\leq \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t)f(s)ds \Big| + \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t)f(t)ds \Big| \\ &= II_A + II_B, \end{aligned}$$

where the equality is due to the definition of II with algebraic manipulation, we employ the triangular inequality in the inequality. We have that

$$\begin{aligned} II_A &= \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t)f(s)ds \Big| \\ &\leq \sup_{T_- \leq t \leq T_+} \sup_{|s|\geq\lambda} \delta_n(s) \int_{|s-t|>\lambda} f(s)ds \\ &\leq \sup_{T_- \leq t \leq T_+} \sup_{|s|\geq\lambda} \delta_n(s) \\ &= \sup_{|s|\geq\lambda} \delta_n(s) \\ &\to 0, \end{aligned}$$

where the first equality comes from the definition of II_A, a sup domination yields the first inequality, we use the fact that f is a pdf in the second inequality, the convergence is obtained by Assumption 2 Expression (16).
As for II_B, we obtain that

$$\begin{aligned} II_B &= \sup_{T_- \leq t \leq T_+} \Big| \int_{|s-t|>\lambda} \delta_n(s-t) f(t) ds \Big| \\ &= \sup_{T_- \leq t \leq T_+} f(t) \int_{|s-t|>\lambda} \delta_n(s-t) ds \\ &= \sup_{T_- \leq t \leq T_+} f(t) \int_{|u|>\lambda} \delta_n(u) du \\ &= C \int_{|u|>\lambda} \delta_n(u) du \\ &\to 0, \end{aligned}$$

where the first equality comes from the definition of II_A, the second equality comes from algebraic manipulation, the third equality is obtained by a change of variable in the Riemmann integral, the fourth equality is due to the fact f is continuous on $[T_-, T_+]$ by Assumption 5 Expression (35) so that f is bounded, and the convergence is a consequence of Assumption 2 Expression (17). □

The next lemma extends Theorem 4 (p. 110) in Watson and Leadbetter (1964b), i.e. convergence of the pdf estimator rescaled variance, to a statement uniform on $[T_-, T_+]$.

Lemma 7. *We assume that Assumption 2, Assumption 3 Expressions (19) and (20), and Assumption 5 Expressions (34) ,(35) and (36) hold. We have*

$$\sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \operatorname{Var}[\widehat{f}_n(t)] - f(t) \Big| \to 0. \tag{88}$$

Proof of Lemma 7. From Equation (4.1) in Watson and Leadbetter (1964b) (p. 110), we can deduce that

$$\frac{n}{\alpha_n} \operatorname{Var}[\widehat{f}_n(t)] = \frac{1}{\alpha_n} \int_{\mathbb{R}} \delta_n^2(t-s) f(s) ds - \frac{1}{\alpha_n} \Big(\int_{\mathbb{R}} \delta_n(t-s) f(s) ds \Big)^2. \tag{89}$$

Then, substituting Equation (89) into Equation (88), we can obtain that

$$\begin{aligned} \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \operatorname{Var}[\widehat{f}_n(t)] - f(t) \Big| &= \sup_{T_- \leq t \leq T_+} \Big| \frac{1}{\alpha_n} \int_{\mathbb{R}} \delta_n^2(t-s) f(s) ds \\ &\quad - \frac{1}{\alpha_n} \Big(\int_{\mathbb{R}} \delta_n(t-s) f(s) ds \Big)^2 - f(t) \Big|. \end{aligned}$$

An use of the triangular inequality along with sup manipulation yields that

$$\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}[\widehat{f}_n(t)]-f(t)\Big| \le \sup_{T_-\le t\le T_+}\Big|\frac{1}{\alpha_n}\int_{\mathbb{R}}\delta_n^2(t-s)f(s)ds-f(t)\Big| + \sup_{T_-\le t\le T_+}\Big|\frac{1}{\alpha_n}\Big(\int_{\mathbb{R}}\delta_n(t-s)f(s)ds\Big)^2\Big|. \tag{90}$$

By Lemma 2 (p. 103) in Watson and Leadbetter (1964b), $\delta_n^2(t)/\alpha_n$ defines a delta-sequence. Thus, we can deduce that the upper term in the right-hand side of Equation (90) goes to 0 by an application of Lemma 6 along with Assumption 2 and Assumption 5 Expressions (34)-(35)-(36). By a direct application of Lemma 6 along with Expression (26), i.e. that $\alpha_n \to \infty$, we can also deduce that the lower term in the right-hand side of Equation (90) goes to 0. □

In the next lemma, we show the convergence uniform on $[T_-, T_+]$ of the rescaled variance of $\widehat{\sigma}^{2,n}_{t,g}$.

Lemma 8. *We assume that Assumptions 1-2-3 and Assumption 5 Expressions (34)-(35)-(36) hold. Then, we have*

$$\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}\big[\widehat{\sigma}^{2,n}_{t,g}\big]-\frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|\to 0.$$

Proof of Lemma 8. We define I as

$$I=\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}\big[\widehat{\sigma}^{2,n}_{t,g}\big]-\frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|.$$

We have that

$$\begin{aligned}
I &= \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}\Big[\frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))}\mathbf{1}_{\{0<\widehat{F}_n(t)<1\}}\Big]- \\
&\quad \frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big| \\
&= \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}\Big[\frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(t)))}\Big]-\frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|+o(1) \\
&= \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}\Big[\frac{\widehat{f}_n(t)}{f_g^W((P_g^W)^{-1}(F(t)))}\Big]-\frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|+o(1) \\
&= \sup_{T_-\le t\le T_+}\frac{1}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|\frac{n}{\alpha_n}\operatorname{Var}[\widehat{f}_n(t)]-f(t)\Big|+o(1) \\
&\le \sup_{T_-\le t\le T_+}\frac{1}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}[\widehat{f}_n(t)]-f(t)\Big|+o(1) \\
&= C\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\operatorname{Var}[\widehat{f}_n(t)]-f(t)\Big|+o(1) \\
&\to 0,
\end{aligned}$$

where the first equality is obtained by Equation (7), the second and third equalities can be deduced following similar arguments as in the proofs of Theorem 1 together with Assumptions 1-2-3 and Assumption 5 Expressions (34)-(35)-(36), the fourth equality is due to the elementary fact that $\mathrm{Var}[cX] = c^2 \mathrm{Var}[X]$ for any random variable X and any deterministic $c \in \mathbb{R}$, the first inequality is due to sup domination, we use the fact that the function $(f_g^W((P_g^W)^{-1}(F(t)))^2$ is continuous and positive by Equation (68) on $[T_-, T_+]$ which is compact by Assumption 5 Expression (34) in the fifth equality, and the convergence is deduced with the application of Lemma 7 along with Assumption 2, Assumption 3 Expressions (19)-(20) and Assumption 5 Expressions (34)-(35)-(36). We have thus shown the lemma. □

In the next lemma, we show the convergence uniform on $[T_-, T_+]$ of the rescaled variance of $\widehat{k(\sigma_{t,g}^2)^n}$.

Lemma 9. *We assume that Assumptions 1, 2 and 3, and Assumption 5 Expressions (34),(35),(36) and (39) hold. Then, we have*

$$\sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \mathrm{Var}\left[\widehat{k(\sigma_{t,g}^2)^n}\right] - \frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| \to 0. \tag{91}$$

Proof of Lemma 9. We define I as

$$I = \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \mathrm{Var}\left[\widehat{k(\sigma_{t,g}^2)^n}\right] - \frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big|.$$

First, we have

$$\begin{aligned} I &= \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \mathrm{Var}\left[k(\widehat{\sigma}_{t,g}^{2,n}) \mathbf{1}_{\{\widehat{\sigma}_{t,g}^{2,n} > 0\}}\right] - \frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| \\ &= \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \mathrm{Var}\left[k(\widehat{\sigma}_{t,g}^{2,n})\right] - \frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| + o(1) \end{aligned} \tag{92}$$

where the first equality comes from Equation (8) and the second equality comes from the assumption that f is positive and continuous on the compact space $[T_-, T_+]$ by Assumption 5 Expressions (34)-(35)-(36). For any $t \in [T_-, T_+]$, we have that

$$\mathrm{Var}[k(\widehat{\sigma}_{t,g}^{2,n})] = \mathrm{Var}[k(\widehat{\sigma}_{t,g}^{2,n}) - k(\sigma_{t,g}^{2,n})], \tag{93}$$

since $\mathrm{Var}[X - c] = \mathrm{Var}[X]$ for any random variable X and any deterministic $c \in \mathbb{R}$. Then, by a Taylor expansion along with Assumption 5 Expression (39) we obtain that

$$k(\widehat{\sigma}_{t,g}^{2,n}) - k(\sigma_{t,g}^2) = k'(\sigma_{t,g}^2)(\widehat{\sigma}_{t,g}^{2,n} - \sigma_{t,g}^2) + \frac{k''(\widetilde{\sigma}_n^2)}{2}(\widehat{\sigma}_{t,g}^{2,n} - \sigma_{t,g}^2)^2, \tag{94}$$

where $\widetilde{\sigma}_n^2$ is between $\widehat{\sigma}_{t,g}^{2,n}$ and $\sigma_{t,g}^2$. We can thus reexpress the variance as

$$\begin{aligned}
\mathrm{Var}[k(\widehat{\sigma}_{t,g}^{2,n})] &= \mathrm{Var}[k'(\sigma_{t,g}^2)(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)+\frac{k''(\widetilde{\sigma}_n^2)}{2}(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)^2] \\
&= \mathrm{Var}[k'(\sigma_{t,g}^2)(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)] \\
&\quad +2\,\mathrm{Cov}[k'(\sigma_{t,g}^2)(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2),\frac{k''(\widetilde{\sigma}_n^2)}{2}(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)^2] \quad (95) \\
&\quad +\mathrm{Var}[\frac{k''(\widetilde{\sigma}_n^2)}{2}(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)^2] \\
&:= I_A+I_B+I_C, \quad (96)
\end{aligned}$$

where we substitute Equation (94) into Equation (93) in the first equality, and use a variance-covariance elementary fact in the second equality. On the one hand, we can reexpress I_A as

$$\begin{aligned}
I_A &= \mathrm{Var}[k'(\sigma_{t,g}^2)(\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2)] \\
&= k'(\sigma_{t,g}^2)^2\,\mathrm{Var}[\widehat{\sigma}_{t,g}^{2,n}-\sigma_{t,g}^2] \\
&= k'(\sigma_{t,g}^2)^2\,\mathrm{Var}[\widehat{\sigma}_{t,g}^{2,n}], \quad (97)
\end{aligned}$$

where the first equality corresponds to the definition of I_A, we use the elementary fact that $\mathrm{Var}[cX]=c^2\,\mathrm{Var}[X]$ for any random variable X and any deterministic $c\in\mathbb{R}$ in the second equality, the third equality is due to the elementary fact that $\mathrm{Var}[X-c]=\mathrm{Var}[X]$. Then, we also have

$$\begin{aligned}
I &= \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}\mathrm{Var}\left[k(\widehat{\sigma}_{t,g}^{2,n})\right]-\frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|+o(1) \\
&= \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}(I_A+I_B+I_C)-\frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|+o(1) \\
&\le \sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}I_A-\frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|+\frac{n}{\alpha_n}|I_B|+\frac{n}{\alpha_n}|I_C|+o(1),
\end{aligned}$$

where the first equality corresponds to Equation (92), we substitute $\mathrm{Var}\left[k(\widehat{\sigma}_{t,g}^{2,n})\right]$ by its value in Equation (96) in the second equality, and the inequality comes from the triangular inequality. We define J as

$$J=\sup_{T_-\le t\le T_+}\Big|\frac{n}{\alpha_n}I_A-\frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2}\Big|.$$

On the one hand, we have that

$$\begin{aligned} J &= \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} k'(\sigma_{t,g}^2)^2 \operatorname{Var}[\widehat{\sigma}_{t,g}^{2,n}] - \frac{k'(\sigma_{t,g}^2)^2 f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| \\ &\leq \sup_{T_- \leq t \leq T_+} k'(\sigma_{t,g}^2)^2 \\ &\quad \times \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \operatorname{Var}[\widehat{\sigma}_{t,g}^{2,n}] - \frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| \\ &= C \sup_{T_- \leq t \leq T_+} \Big| \frac{n}{\alpha_n} \operatorname{Var}[\widehat{\sigma}_{t,g}^{2,n}] - \frac{f(t)}{(f_g^W((P_g^W)^{-1}(F(t)))^2} \Big| \\ &\to 0, \end{aligned}$$

where we substitute I_A by its value in Equation (97) in the first equality, the first inequality is due to sup domination, the second equality is implied by the fact that k' is continuous on

$$\Big[\inf_{T_- \leq t \leq T_+} \sigma_{t,g}^2, \sup_{T_- \leq t \leq T_+} \sigma_{t,g}^2\Big]$$

given Assumption 5 Expression (39) and by the fact that the instantaneous variance $\sigma_{t,g}^2$ is bounded on $[T_-, T_+]$ by Equation (4), and the convergence comes from Lemma 8 along with Assumptions 1-2-3 and Assumption 5 Expressions (34),(35) and (36). On the other hand, we can show that $\frac{n}{\alpha_n}|II| \to 0$ and $\frac{n}{\alpha_n}|III| \to 0$ using similar proving techniques. We have thus shown Equation (91). □

We show the convergence of the rescaled variance of $\widehat{C}_g^n$ in the next proposition. The asymptotic variance of $\widehat{C}_g^n$ is equal to the cumulative asymptotic variance of $\widehat{k(\sigma_{t,g}^2)}^n$.

Proposition 2. *We assume that Assumptions 1, 2 and 3, and Assumption 5 Expressions (33), (34), (35), (36) and (39) hold. Then, we have*

$$\frac{n}{\alpha_n} \operatorname{Var}[\widehat{C}_g^n] \to \int_{T_-}^{T_+} \frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt \tag{98}$$

as $n \to \infty$.

Proof of Proposition 2. We have that

$$\begin{aligned} \frac{n}{\alpha_n} \operatorname{Var}\big[\widehat{C}_g^n\big] &= \frac{n}{\alpha_n} \operatorname{Var}\Big[\sum_{l=0}^{M_n-1} \widehat{k(\sigma_{T_l^n,g}^2)}^n \Delta_n\Big] \\ &= \frac{n}{\alpha_n} \sum_{l=0}^{M_n-1} \operatorname{Var}\big[\widehat{k(\sigma_{T_l^n,g}^2)}^n \Delta_n\big] \\ &\quad + \frac{n}{\alpha_n} \sum_{m,l=0 \text{ s.t. } m \neq l}^{M_n-1} \operatorname{Cov}[\widehat{k(\sigma_{T_m^n,f}^2)}^n \Delta_n, \widehat{k(\sigma_{T_l^n,f}^2)}^n \Delta_n] \\ &:= I + II, \end{aligned}$$

where the first equality is obtained by Equation (11), the second equality is obtained by a variance-covariance elementary fact and algebraic manipulation. On the one hand, we have that

$$\begin{aligned} I &= \frac{n}{\alpha_n}\sum_{l=0}^{M_n-1} \mathrm{Var}\left[k(\widehat{\sigma^2_{T_l^n,g}})^n \Delta_n\right] \\ &= \sum_{l=0}^{M_n-1} \frac{k'(\sigma^2_{T_l^n,g})^2 f(T_l^n)}{f_g^W((P_g^W)^{-1}(F(T_l^n)))^2}\Delta_n + o(1) \\ &\to \int_{T_-}^{T_+} \frac{k'(\sigma^2_{t,g})^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt, \end{aligned}$$

where the first equality corresponds to the definition of I, the second equality comes from Lemma 9 along with Assumptions 1, 2 and 3 and Assumption 5 Expressions (34),(35),(36) and (39), and we use Riemann integral definition along with the fact that the integrated function is continuous on $[T_-, T_+]$ and Assumption 5 Expression (33) for the convergence. On the other hand, we have that

$$\begin{aligned} II &= \frac{n}{\alpha_n}\sum_{m,l=0 \text{ s.t. } m\neq l}^{M_n-1} \mathrm{Cov}[k(\widehat{\sigma^2_{T_m^n,f}})^n \Delta_n, k(\widehat{\sigma^2_{T_l^n,f}})^n \Delta_n] \\ &= \frac{1}{\alpha_n} O(1) \\ &\to 0, \end{aligned}$$

where the first equality corresponds to the definition of II, the second equality is a consequence to the fact that $n\,\mathrm{Cov}[\widehat{f}_n(t), \widehat{f}_n(s)] \to -f(t)f(s)$ for any $t \neq s$ from Watson and Leadbetter (1964b) (Section 4, p. 110) along with Assumption 2, Assumption 3 Expressions (19)-(20) and Assumption 5 Expressions (35)-(36), and the convergence is obtained as $\alpha_n \to \infty$ by Expression (26). □

In what follows, we give the proof of Theorem 3, which in particular extends the arguments from the proof of Theorem 5 (pp. 111-112) in Watson and Leadbetter (1964b) based on CLT normal convergence criterion (p. 307) from Loeve (1977) along with the use of Proposition 2.

Proof of Theorem 3. We have that

$$\begin{aligned}
\sqrt{\frac{n}{\alpha_n}}(\widehat{C}_g^n - C_g) &= \sqrt{\frac{n}{\alpha_n}}\Big(\sum_{l=0}^{M_n-1} k(\widehat{\sigma^2_{T_l^n,g}})^n \Delta_n - \int_{T_-}^{T_+} k(\sigma^2_{t,g})dt\Big) \\
&= \sqrt{\frac{n}{\alpha_n}}\Big(\sum_{l=0}^{M_n-1} k(\widehat{\sigma^2_{T_l^n,g}})^n \Delta_n - \int_{T_l^n}^{T_{l+1}^n} k(\sigma^2_{t,g})dt\Big)\Big) \\
&= \sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1} \Big(k(\widehat{\sigma^2_{T_l^n,g}})^n - \mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n]\Big)\Delta_n \\
&\quad + \sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1} \Big(\mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n]\Delta_n - \int_{T_l^n}^{T_{l+1}^n} k(\sigma^2_{t,g})dt\Big) \\
&= I + II,
\end{aligned}$$

where we use Equation (11) and Equation (9) in the first equality, and the second and third equalities correspond to algebraic manipulation. In what follows, we will show that

$$I \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, (T_+ - T_-)\int_{T_-}^{T_+} \frac{k'(\sigma^2_{t,g})^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt\Big), \tag{99}$$

$$II \xrightarrow{\mathbb{P}} 0. \tag{100}$$

We can prove Expression (99) extending the arguments from the proof of Theorem 5 (pp. 111-112) in Watson and Leadbetter (1964b), which is based on CLT assumptions from Loeve (1977) along with the use of Proposition 2, Assumptions 1-2-3 and Assumption 5 Expressions (33)-(34)-(35)-(36)-(39).

We now show that Expression (100) holds. We have that

$$\begin{aligned}
\mid II \mid &= \sqrt{\frac{n}{\alpha_n}}\Big|\sum_{l=0}^{M_n-1}\Big(\mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n]\Delta_n - \int_{T_k^n}^{T_{k+1}^n} k(\sigma^2_{t,g})dt\Big)\Big| \\
&= \sqrt{\frac{n}{\alpha_n}}\Big|\sum_{l=0}^{M_n-1}\Big(\mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n] - k(\sigma^2_{T_l^n,f})\Big)\Delta_n \\
&\quad + \sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1}\Big(k(\sigma^2_{T_l^n,g})\Delta_n - \int_{T_l^n}^{T_{k+1}^n} k(\sigma^2_{t,g})dt\Big)\Big| \\
&\leq \Big|\sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1}\Big(\mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n] - k(\sigma^2_{T_l^n,g})\Big)\Delta_n\Big| + \\
&\quad \Big|\sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1}\Big(k(\sigma^2_{T_l^n,g})\Delta_n - \int_{T_l^n}^{T_{l+1}^n} k(\sigma^2_{t,g})dt\Big)\Big| \\
&\leq \Big|\sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1}\Big(\mathbb{E}[k(\widehat{\sigma^2_{T_l^n,g}})^n] - k(\sigma^2_{T_l^n,g})\Big)\Delta_n\Big| \\
&\quad + \sqrt{\frac{n}{\alpha_n}}\sum_{l=0}^{M_n-1}\int_{T_l^n}^{T_{l+1}^n}\Big|k(\sigma^2_{T_l^n,f}) - k(\sigma^2_{t,g})\Big|dt \\
&= II_A + II_B,
\end{aligned}$$

where the first equality is obtained by the definition of II, the second equality corresponds to algebraic manipulation, and the two inequalities are due to the use of triangular inequalities. On the one hand, we have that $II_A \to 0$ can be obtained extending the arguments to the bias case in the proof of Lemma 9 along with Assumptions 1, 2 and 3, and Assumption 5 Expressions (33),(34),(35),(36),(37),(38) and (39).

On the other hand, by a Taylor expansion to the function $H(t) = k(\sigma^2_{t,g})$ along with Assumption 5 Expressions (35)-(39) we obtain that

$$k(\sigma^2_{t,g}) - k(\sigma^2_{T_l^n,g}) \quad = \quad H'(t_l)(t - T_l^n), \tag{101}$$

where $T_l^n \leq t_l \leq t$. Then, we have that

$$\begin{aligned}
II_B &= \sqrt{\frac{n}{\alpha_n}} \sum_{l=0}^{M_n-1} \int_{T_l^n}^{T_{l+1}^n} \Big| k(\sigma^2_{T_l^n,g}) - k(\sigma^2_{t,g}) \Big| dt \\
&= \sqrt{\frac{n}{\alpha_n}} \sum_{l=0}^{M_n-1} \int_{T_l^n}^{T_{l+1}^n} \Big| H'(t_l)(t - T_l^n) \Big| dt \\
&\leq \sqrt{\frac{n}{\alpha_n}} \sup_{T_- \leq t \leq T_+} \Big| H'(t) \Big| \sum_{l=0}^{M_n-1} \int_{T_l^n}^{T_{l+1}^n} \Big| t - T_l^n \Big| dt \\
&= \sqrt{\frac{n}{\alpha_n}} C \sum_{l=0}^{M_n-1} \int_{T_l^n}^{T_{l+1}^n} \Big| t - T_l^n \Big| dt \\
&= \sqrt{\frac{n}{\alpha_n}} C M_n \frac{\Delta_n^2}{2} \\
&= \frac{C(T_+ - T_-)^2}{2} \sqrt{\frac{n}{\alpha_n}} \frac{1}{M_n} \\
&\to 0,
\end{aligned}$$

where the first equality corresponds to the definition of II_B, the second equality comes from Equation (101), the first inequality is obtained by sup domination, the third equality is a consequence to the fact that h' is continuous on $[T_-, T_+]$ by Assumption 5 Expression (35) and Equation (4), the fourth equality is obtained by an integral calculation and algebraic manipulation, we use Equation (10) in the fifth equality, the convergence is deduced by Assumption 5 Expression (33). Thus, we have shown the CLT (41). It remains to prove the feasible CLT (42). First, we obtain the following consistency, i.e.

$$\sum_{l=0}^{M_n-1} \frac{k'(\widehat{\sigma}^2_{T_l^n,g})^2 \widehat{f}_n(T_l^n)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(T_l^n)))^2} \Delta_n \xrightarrow{\mathbb{P}} \int_{T_-}^{T_+} \frac{k'(\sigma^2_{t,g})^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt$$

by extending the arguments in the proofs of Lemma 4 and Lemma 5 to obtain a uniform convergence and using Lemma 3 and Assumption 5 Expressions (33)-(39). We define I_n as

$$I_n = \sqrt{\Big(\sum_{l=0}^{M_n-1} \frac{k'(\widehat{\sigma}^2_{T_l^n,g})^2 \widehat{f}_n(T_l^n)}{f_g^W((P_g^W)^{-1}(\widehat{F}_n(T_l^n)))^2} \Delta_n \Big)^{-1}}.$$

Then, we can deduce by the continuous mapping theorem that

$$I_n \xrightarrow{\mathbb{P}} \sqrt{\Big(\int_{T_-}^{T_+} \frac{k'(\sigma^2_{t,g})^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt \Big)^{-1}}. \tag{102}$$

By Slutsky's lemma along with Equation (102) and the CLT (41), we obtain that

$$\sqrt{\frac{n}{\alpha_n}} I_n(\widehat{C}_g^n - C_g) \quad \overset{\mathcal{D}}{\to} \quad \sqrt{\Big(\int_{T_-}^{T_+} \frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt\Big)^{-1}} \times \mathcal{N}\Big(0, \int_{T_-}^{T_+} \frac{k'(\sigma_{t,g}^2)^2 f(t)}{f_g^W((P_g^W)^{-1}(F(t)))^2} dt\Big).$$

Finally, we can deduce the feasible CLT (42) noting that $a\mathcal{N}(0,b) = \mathcal{N}(0, a^2 b)$ for any $a > 0$ and $b > 0$. □

In what follows, we give the proof of Proposition 1 which extends the arguments in the proof of Theorem 3 since boundary tuning parameter estimation requires uniform arguments on g.

Proof of Proposition 1. We have that for any fixed $t \geq 0$, the hitting-time variance is a strictly increasing function of g by Equation (51) and solution of the IFHT problem by Assumption 1. Thus, the function $g \to \frac{C_g(T_-,T_+)}{T_+ - T_-}$ is a strictly increasing function of g since $0 < T_- < T_+$ and by Assumption 6. Then, we can deduce the existence and uniqueness of $\overline{g} > 0$ which satisfies Equation (12) by the intermediate value theorem. We also have that for any fixed $t \geq 0$, the hitting-time variance estimator is a strictly increasing function of g by Equation (7). Thus, the function $g \to \frac{\widehat{C}_g^n(T_-,t)}{t - T_-}$ is a strictly increasing function by the fact that $0 < T_- < t$, Equation (11) and Assumption 6. Then, we can deduce the existence and uniqueness of $\widehat{g}_n > 0$ which satisfies Equations (13)-(14) by the intermediate value theorem. We prove now the consistency, i.e. $\widehat{g}_n \overset{\mathbb{P}}{\to} \overline{g}$. Since the function $g \to C_g(T_-,T_+)$ is continuous and positive in $\overline{g}$, it is sufficient to show that $\big|C_{\widehat{g}_n}(T_-,T_+) - C_{\overline{g}}(T_-,T_+)\big| = o_{\mathbb{P}}(1)$. We have

$$\begin{aligned} \big|C_{\widehat{g}_n}(T_-,T_+) - C_{\overline{g}}(T_-,T_+)\big| &= \big|C_{\widehat{g}_n}(T_-,T_+) - \widehat{C}_{\widehat{g}_n}^n(T_-,T_+) \\ &\quad + \widehat{C}_{\widehat{g}_n}^n(T_-,T_+) - C_{\overline{g}}(T_-,T_+)\big| \\ &\leq \big|C_{\widehat{g}_n}(T_-,T_+) - \widehat{C}_{\widehat{g}_n}^n(T_-,T_+)\big| \\ &\quad + \big|\widehat{C}_{\widehat{g}_n}^n(T_-,T_+) - C_{\overline{g}}(T_-,T_+)\big| \\ &= \big|C_{\widehat{g}_n}(T_-,T_+) - \widehat{C}_{\widehat{g}_n}^n(T_-,T_+)\big| + o_{\mathbb{P}}(1) \end{aligned}$$

where we use the triangular inequality in the inequality, and Equations (12)-(13)-(14) along with Assumption 5 Expression (36) in the second equality. We have

$$\begin{aligned} \big|C_{\widehat{g}_n}(T_-,T_+) - \widehat{C}_{\widehat{g}_n}^n(T_-,T_+)\big| &\leq \sup_{g \in [g_-, g_+]} \big|C_g(T_-,T_+) - \widehat{C}_g^n(T_-,T_+)\big| + o_{\mathbb{P}}(1) \\ &= o_{\mathbb{P}}(1). \end{aligned}$$

where we define $g_- \in \mathbb{R}_*^+$ and $g_+ \in \mathbb{R}_*^+$ such that they satisfy $0 < g_- < g_+$ and we use Assumption 5 Expression (36) in the inequality, we extend the arguments in the proof of Theorem 3 with uniformity on $g \in [g_-, g_+]$ along with Assumptions 1-2-3 and Assumption 5 Expressions (33)-(34)-(35)-(36)-(37)-(38)-(39). □

We give the proof of Corollary 1 in what follows.

Proof of Corollary 1. This is a consequence of Proposition 1, Theorem 3 along with Slutsky's theorem. □

Finally, we give the proof of Lemma 1, which is based on Doob (1949) (Formula (4.2), p. 397) and Malmquist (1954) (Theorem 1, p. 526).

Proof of Lemma 1. Based on Doob (1949) (Formula (4.2), p. 397), Malmquist (1954) (Theorem 1, p. 526) provides an extension where the probability that a standard Brownian motion stays below a linear boundary $g(t) = at + b$ with slope a and intercept b conditioned on its arrival value s at arrival time $T > 0$ is given by

$$\mathbb{P}(W_t \leq at + b \forall t \in [0,T] | W_T = s) = 1 - e^{-2ab + \frac{2b(s-b)}{T}} \text{ for any } s \leq aT + b. \quad (103)$$

Wang and Pötzelberger (1997) (Equation (3), p. 55) integrate Equation (103) with respect to the Brownian motion arrival value $W_T = s$, and obtain that the cdf satisfies $F(0) = 0$,

$$F(t) = 1 - \Phi\left(\frac{at+b}{\sqrt{t}}\right) + e^{-2ab}\Phi\left(\frac{at-b}{\sqrt{t}}\right) \text{ for any } t > 0.$$

Thus, we have shown Equation (64). Then, we can deduce the pdf for any $t > 0$ as

$$\begin{aligned}
f(t) &= \frac{d}{dt}F(t) \\
&= \frac{d}{dt}\left(1 - \Phi\left(\frac{at+b}{\sqrt{t}}\right) + e^{-2ab}\Phi\left(\frac{at-b}{\sqrt{t}}\right)\right) \\
&= \frac{d}{dt}\left(1 - \int_{-\infty}^{\frac{at+b}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + e^{-2ab} \int_{-\infty}^{\frac{at-b}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right) \\
&= -\frac{ae^{-\frac{(at+b)^2}{2t}}}{2\sqrt{2\pi t}} + \frac{be^{-\frac{(at+b)^2}{2t}}}{2\sqrt{2\pi t^3}} + e^{-2ab}\left(\frac{ae^{-\frac{(at-b)^2}{2t}}}{2\sqrt{2\pi t}} + \frac{be^{-\frac{(at-b)^2}{2t}}}{2\sqrt{2\pi t^3}}\right) \\
&= \frac{be^{-\frac{(at+b)^2}{2t}}}{\sqrt{2\pi t^3}},
\end{aligned}$$

where the fact that f is the density of F in the first equality, the second equality comes from Equation (64), the third equality corresponds to Equation (59), the fourth equality is obtained with the fundamental theorem of calculus along with the chain rule and the fifth equality is obtained by algebraic manipulation. When $t \to 0$, we obtain that $f(t) \to 0$ and we can deduce that $f(0) = 0$ by continuity. We have thus proven Equation (63). □