Homework 5 of *Stochastic Calculus, An introduction with Application*

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Due: Thursday, July 2020 23rd 23:59pm by email to potiron@fbc.keio.ac.jp

We consider an emblematic problem in financial econometrics for high-frequency data, i.e. estimating volatility. We assume that the underlying efficient price process is defined as $X_t = mt + \sigma B_t$ on $[0, 1]$ where B_t is a standard Brownian motion, $m \in \mathbb{R}$ is the drift parameter, and we aim to estimate the variance parameter σ^2. Note that $[0, 1]$ stands for the economic day of interest.

1 Usual framework from the class

In this part, we assume that we observe the price process X_t at regular times $t_{0,n} = 0, t_{1,n} = \frac{1}{n}, t_{2,n} = \frac{2}{n}, \cdots, t_{n,n} = 1$.

1. Recall the definition of the Realized Volatility (RV) estimator $\widehat{\sigma}_n^2$. Compute the bias and the variance of $\widehat{\sigma}_n^2$. Interpret about those results when n goes to infinity.
2. Show that $\mathbb{E}\big[(\widehat{\sigma}_n^2 - \sigma^2)^2\big] \to 0$.

2 Non regular observations

1. Redo (as much as you can from) Part 1 assuming that X_t is observed at deterministic non regular times $t_{0,n} = 0, t_{1,n} = f(\frac{1}{n}), t_{2,n} = f(\frac{2}{n}), \cdots, t_{n,n} = f(1)$. where

$f : [0,1] \to [0,1]$ is a strictly increasing smooth function such that $f(0) = 0$ and $f(1) = 1$.

2. Redo (as much as you can from) Part 1 assuming that the observation times $t_{0,n}, \cdots, t_{N,n}$ where N is random and $t_{i,n}$, which is independent from X_t, follows a Poisson point process with parameter $n\mu$ with $\mu > 0$, i.e. $\Delta t_i := t_i - t_{i-1}$ are i.i.d and follows an exponential distribution with parameter $n\mu$.

3 Noisy observations

1. We assume now that we observe a noisy version of the price process, i.e that we observe $Z_t = X_t + n^\alpha \epsilon_t$, where ϵ_t are i.i.d random variables distributed as $\mathcal{N}(0, \sigma_\epsilon^2)$ and $0 \geq \alpha$, but that the observation times are regular (as in Part 1). Moreover, we assume that ϵ_t is independent from X_t. Find the natural definition of $\widehat{\sigma}_n^2(Z)$ in that case by replacing X_t by Z_t in the original definition. Compute the bias and the variance of $\widehat{\sigma}_n^2(Z)$. What happens when n goes to infinity ? Discuss the different cases related to the choice of the value α.

2. Discuss question 2 of Part 1 in that new case, i.e. calculate $\mathbb{E}\big[(\widehat{\sigma}_n^2(Z) - \sigma^2)^2\big]$. What is the limit depending on α?