# Homework 2 of *Stochastic Calculus, An introduction with Application*

Keio University

Due: Thursday June 2020 4th 23:59pm

How: by sending an email (with Object: [S2020H2] LAST_NAME FIRST_NAME) to potiron@fbc.keio.ac.jp after being converted to PDF format (with the format LastNameFirstName.pdf in Romaji/English) and not handwritten.

1. We recall that for $X$ any random variable, we have:

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Show that

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

2. It is often the case that we don't know the parameters of the normal distribution, but instead want to estimate them. That is, observing $(X_1, X_2, \cdots, X_n)$ from $n$ independent random variables normally distributed $\mathcal{N}(\mu, \sigma^2)$ we would like to learn the approximate values of parameters $\mu$ and $\sigma^2$.

   (a) We estimate $\mu$ with the sample mean as

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

   i. Calculate the bias of $\hat{\mu}_n$, i.e. calculate $B_n := \mathbb{E}[(\hat{\mu}_n - \mu)]$.
   ii. Calculate the limit of $B_n$ as $n \to \infty$, and interpret the result.
   iii. Calculate the variance of $\hat{\mu}_n$, i.e. calculate $V_n := \mathrm{Var}[\hat{\mu}_n]$.

iv. Calculate the limit of $V_n$ as $n \to \infty$, and interpret the result.

(b) We estimate $\sigma^2$ as

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu}_n)^2.$$

Redo all the questions of the previous part with this estimator.