Fixpoint theorem in linear set theory

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Abstract

In this paper, we first show that the fixpoint term can be constructed for any formula in the system of linear set theory with equality and pairing. We then prove that all the total recursive functions are numeralwise representable in such a system. Furthermore, we observe that the additive infinitary extension of the system would become inconsistent if the extensionality principle were added.

1 Introduction

In this paper, we launch the investigation into the expressive power of the set theory based on contraction-free logics. Such a set theory is initially introduced by Grishin with the observation that the principle of unrestricted comprehension formulated as the inference rules

$$\frac{\Gamma, A[s/x] \vdash \Delta}{\Gamma, s \in \{x : A\} \vdash \Delta} \qquad \frac{\Gamma \vdash A[s/x], \Delta}{\Gamma \vdash s \in \{x : A\}, \Delta}$$

is available without the standard set-theoretical paradoxes.

Terms constructed through the unrestricted comprehension are very similar to those of the untyped lambda-calculus. In particular, we show that the familiar fixpoint construction for the untyped lambda calculus can be modified so that that the similar fixpoint result can be obtained for the set theory based on contraction-free logics.

The fixpoint construction will be used to study the expressive power of the set theory based on contraction-free logics. Since the consistency of Grishin's system is established by the ordinary induction (*i.e.* up to ω), its expressive power is much weaker than the Peano Arithmetic. We show, however, that Grishin's system is at least as strong as to obtain the numeralwise representability of all the total recursive functions.

The availability of fixpoints in predicate linear logic has already been noticed by White, Peter Schröder-Heister and J.Y.Girard. However, they introduced fixpoints as the primitive construct and showed that their systems are consistent by the cut-elimination argument. What we show in this paper is that the fixpoints can be explicitly constructed from the principle of unrestricted comprehension in contraction-free logics.

2 Linear set theory with equality and pairing

We will work in the first-order two-sided sequent calculi for linear and affine logics enhanced with the principle of unrestricted comprehension and the equality axioms.

The terms t and formulas A of our system are freely generated from the fixed infinite set Var of variables v as given in the following BNF:

$$\begin{array}{rcl} v & \in & Var \\ t & ::= & v \mid \{x : A\} \\ A & ::= & t = t \mid t \neq t \mid t \in t \mid t \notin t \mid A \otimes A \mid A \parr A \mid A \& A \mid A \oplus A \mid \forall v\, A \mid \exists v\, A \end{array}$$

In addition the duals $A^{\perp}$ for formulas A are defined by $(s \in t)^{\perp} \equiv s \notin t$, $(s \notin t)^{\perp} \equiv s \in t$, $(s = t)^{\perp} \equiv s \neq t$ and $(s \neq t)^{\perp} \equiv s = t$ for atomic formulas, and the De Morgan laws. The axioms and rules of inference are those of the first-order MALL fragment of linear and affine logics (linear logic and weakening) with the following extras:

- the principle of unrestricted comprehension

$$(\in\mathrm{L})\ \frac{\Gamma, A[s/x] \vdash \Delta}{\Gamma, s \in \{x : A\} \vdash \Delta} \qquad\qquad (\in\mathrm{R})\ \frac{\Gamma \vdash A[s/x], \Delta}{\Gamma \vdash s \in \{x : A\}, \Delta}$$

- the axioms for equality

$$(\mathrm{Id})\ \vdash s = s \qquad\qquad (\mathrm{Sub})\ s = t, A \vdash A[t/s].$$

  where $A[t/s]$ stands for the formula obtained from the formula A by substituting some of the occurrences of the term s by the term t.

The equality relation is rather special in the setting of contraction-free logics, since one can derive the left contraction rule for the atomic formulas $s = t$ as seen below:

$$\frac{\dfrac{\vdash s = s \quad \vdash s = s}{\vdash s = s \otimes s = s} \qquad s = t, s = s \otimes s = s \vdash s = t \otimes s = t}{s = t \vdash s = t \otimes s = t}$$

In fact, the equality relation together with the naively formulated extensionality principle

$$\frac{\Gamma, x \in s \vdash x \in t, \Delta \quad \Pi, x \in t \vdash x \in s, \Sigma}{\Gamma, \Pi \vdash s = t, \Delta, \Sigma}$$

yields the contraction for all the formulas and the paradox as its consequence[4]. On the other hand, the equality relation itself can be defined through the Leibniz equality

$$\forall x(s \in x \multimap t \in x) \& \forall x(t \in x \multimap s \in x)$$

so that the standard axioms of the equality are satisfied[2]:

$$\dfrac{\dfrac{\dfrac{s \in x \vdash s \in x}{\vdash s \in x \multimap s \in x}}{\vdash \forall x(s \in x \multimap s \in x)} \quad \dfrac{\dfrac{s \in x \vdash s \in x}{\vdash s \in x \multimap s \in x}}{\vdash \forall x(s \in x \multimap s \in x)}}{\vdash s = s}$$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash s \in \{x : A[x/s]\}} \quad \dfrac{\forall x(s \in x \multimap t \in x), s \in \{x : A[x/s]\} \vdash t \in \{x : A[x/s]\}}{s = t, s \in \{x : A[x/s]\} \vdash t \in \{x : A[x/s]\}}}{s = t, A \vdash t \in \{x : A[x/s]\}} \quad \dfrac{A[t/s] \vdash A[t/s]}{t \in \{x : A[x/s]\} \vdash A[t/s]}}{s = t, A \vdash A[t/s]}$$

This shows that our system is a conservative extension of the system without the equality relation, which is essentially Grishin's original one.

We remain intentionally ambiguous about the choice of linear and affine logics. For the results stated in this section, it is sufficient to work in linear logic. The representability result in the later section, however, requires the heavy use of weakening, i.e. affine logic.

Regardless whether we have weakening or not, the consistency of the system can be shown by the cut-elimination method with the induction up to ω. Since this is already a well-established technique, we only give the reference to the literature [3, 8, 9, 11].

Once we have the equality relation, the ordered pairs $\langle s, t\rangle$ can be defined in linear set theory by the standard construction $\{\{s\}, \{s, t\}\}$, where the unordered pairs $\{s, t\}$ are defined by using the additive disjunction as $\{x : x = s \oplus x = t\}$. Note that the proof by cases is not possible if we use the multiplicative disjunction instead of the additive one.

Under this definition, we can show the standard property of the ordered pairs. First, let's write $!_n\, A$ for

$$\underbrace{A \otimes A \otimes \ldots \otimes A}_{n}.$$

We can then obtain the necessary properties of the ordered pairs without using any contraction.

Lemma 1. *The following sequents are provable:*

1. $s = s' \otimes t = t' \vdash \langle s, t\rangle = \langle s', t'\rangle$,
2. $\langle s, t\rangle = \langle s', t'\rangle \vdash s = s'$,
3. $s = s' \otimes\, !_2\, \langle s, t\rangle = \langle s', t'\rangle \vdash t = t'$,
4. $!_3\, \langle s, t\rangle = \langle s, t'\rangle \vdash t = t'$.

Proof In the proofs of this paper, we often use the informal argument to establish the provability of sequents, from which the derivations in the sequent calculus can easily be reconstructed.

1. We have $\vdash \langle s, t\rangle = \langle s, t\rangle$ by the axiom (Id). Using the cuts with the axioms (Sub) twice, we obtain the desired sequent.
2. We have $\vdash \{s\} \in \langle s, t\rangle$. Hence $\langle s, t\rangle = \langle s', t'\rangle \vdash \{s\} \in \langle s', t'\rangle$. Then

$$\langle s, t\rangle = \langle s', t'\rangle \vdash \{s\} = \{s'\} \oplus \{s\} = \{s', t'\}.$$

Now $\vdash s \in \{s\}$ so that $\{s\} = \{s'\} \vdash s \in \{s'\}$. It follows that $\{s\} = \{s'\} \vdash s = s'$. On the other hand, $\vdash s' \in \{s', t'\}$. Hence $\{s\} = \{s', t'\} \vdash s' \in \{s\}$. It follows that $\{s\} = \{s', t'\} \vdash s = s'$. Hence

$$\{s\} = \{s'\} \oplus \{s\} = \{s', t'\} \vdash s = s'.$$

Using cut, we obtain the desired sequent.

3. We have $\{s, t\} \in \langle s, t\rangle$ and $\{s', t'\} \in \langle s', t'\rangle$. Then

$$\begin{array}{rcl} \langle s, t\rangle = \langle s', t'\rangle & \vdash & \{s, t\} = \{s'\} \oplus \{s, t\} = \{s', t'\} \\ \langle s, t\rangle = \langle s', t'\rangle & \vdash & \{s', t'\} = \{s\} \oplus \{s', t'\} = \{s, t\}. \end{array}$$

Since $\vdash t \in \{s, t\}$, we have $\{s, t\} = \{s'\} \vdash t = s'$ and $\{s, t\} = \{s', t'\} \vdash t = s' \oplus t = t'$. Using ($\oplus$R) and ($\oplus$L),

$$\{s, t\} = \{s'\} \oplus \{s, t\} = \{s', t'\} \vdash t = s' \oplus t = t'.$$

Similarly,

$$\{s', t'\} = \{s\} \oplus \{s', t'\} = \{s, t\} \vdash t' = s \oplus t' = t.$$

Now $s = s', t = s', t' = s \vdash t = t'$ and $s = s', t = t', t' = s \vdash t = t'$. Hence

$$s = s', t = s' \oplus t = t', t' = s \vdash t = t'.$$

On the other hand, $s = s', t = s', t' = t \vdash t = t'$ and $s = s', t = t', t' = t \vdash t = t'$. so that

$$s = s', t = s' \oplus t = t', t' = t \vdash t = t'.$$

Therefore,

$$s = s', t = s' \oplus t = t', t' = s \oplus t' = t \vdash t = t'.$$

Using cut, we obtain the desired sequent.

4. From 2 and 3, we obtain the desired sequent by cut.

∎

With the contraction for $\langle s, t\rangle = \langle s', t'\rangle$ always available, we can obtain the standard property of the ordered pairs.

3 Fixpoint theorem

We show that one can construct a fixpoint term f for any formula A in the sense that $x \in f$ is logically equivalent to $A[f/y]$, in any linear set theory with equality and ordered pairs. Intuitively, terms s can be regarded as functions $t \mapsto t \in s$ and the formulas $A(x, y)$ are functionals $s \mapsto A(x, s)$. Our fixpoints are those for such functionals. In this section, the substitutions should be understood as the substitution of *all the occurrences* of the relevant variables.

Theorem 2. *For any formula A, there exists a term f such that*

$$x \in f \vdash A[f/y] \qquad\qquad A[f/y] \vdash x \in f$$

are both derivable.

Proof Let u, v and w be fresh variables, and define

$$\begin{array}{rcl} s & \equiv & \{z : \exists u \exists v(z = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])\} \\ f & \equiv & \{w : \langle w, s\rangle \in s\} \end{array}$$

First, we have $x \in f \vdash \langle x, s\rangle \in s$. Hence,

$$x \in f \vdash \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x]).$$

From this, it follows that

$$x \in f \vdash A[\{w : \langle w, s\rangle \in s\}/y]$$

i.e., $x \in f \vdash A[f/y]$.

Secondly,

$$A[f/y] \vdash \langle x, s\rangle = \langle x, s\rangle \otimes A[\{w : \langle w, s\rangle \in s\}/y]$$

so that

$$A[f/y] \vdash \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x]).$$

Therefore,

$$A[f/y] \vdash \langle x, s\rangle \in s$$

and

$$A[f/y] \vdash x \in \{w : \langle w, s\rangle \in s\}$$

i.e., $A[f/y] \vdash x \in f$. ∎

4 Numeralwise representability

We now consider the numeralwise representation of arithmetic in linear set theory. In this section, we always asuume *weakening*.

Numerals The numerals $\mathbf{n}$ for the natural numbers n are defined in the standard way. We write $\cup s$ for $\{x : \exists y(x \in y \otimes y \in s)\}$ and $s \cup t$ for $\cup\{s, t\}$. The successor operation *suc* is defined as $sc(v) \equiv s \cup \{s\}$. Then, $\mathbf{0} \equiv \{x : x \neq x\}$ and $\mathbf{n+1} \equiv sc(\mathbf{n})$. We write $\emptyset$ for the empty multiset. This is useful for the proof by contradiction, and we can delete the contradictory cases as follows.

$$\dfrac{\dfrac{A_1 \vdash \emptyset}{A_1 \vdash B} \quad A_2 \vdash B}{A_1 \oplus A_2 \vdash B}$$

where we use weakening with the formula B.

Proposition 3. *The following are provable for any natural numbers n and m:*

1. $\mathbf{n} \in \mathbf{m} \vdash \mathbf{n} \subset \mathbf{m}$,
2. $\mathbf{n} \in \mathbf{m} \otimes \mathbf{m} \in \mathbf{p} \vdash \mathbf{n} \in \mathbf{p}$,
3. $\mathbf{n} \in \mathbf{n} \vdash \emptyset$,

4. $\mathbf{n} = \mathbf{m} \vdash \emptyset$, *for* $n \neq m$.

Proof

1. We use the induction on m. When $m = 0$, this holds vacuously. Suppose $\mathbf{n} \in \mathbf{p} + \mathbf{1}$. Then, $\mathbf{n} \in \mathbf{p}$ or (in the sense of $\oplus$) $\mathbf{n} = \mathbf{p}$. In the former case, $\mathbf{n} \subset \mathbf{p} \subset \mathbf{p} + \mathbf{1}$ by the inductive hypothesis. In the latter case, $\mathbf{n} = \mathbf{p} \subset \mathbf{p} + \mathbf{1}$.

2. Suppose $\mathbf{n} \in \mathbf{m} \otimes \mathbf{m} \in \mathbf{p}$. Then, $\mathbf{n} \in \mathbf{m} \otimes \mathbf{m} \subset \mathbf{p}$. Hence, $\mathbf{n} \in \mathbf{p}$.

3. We use the induction on n. When $n = 0$, this is obvious. Suppose $\mathbf{p} + \mathbf{1} \in \mathbf{p} + \mathbf{1}$. Then, $\mathbf{p} + \mathbf{1} \in \mathbf{p}$ or $\mathbf{p} + \mathbf{1} = \mathbf{p}$. In the former case, $\mathbf{p} \in \mathbf{p} + \mathbf{1} \subset \mathbf{p}$. Hence, $\mathbf{p} \in \mathbf{p}$. By the inductive hypothesis, we have then $\emptyset$. In the latter case, $\mathbf{p} \in \mathbf{p} + \mathbf{1} = \mathbf{p}$. Hence $p \in p$ so that we have again $\emptyset$.

4. Assume $\mathbf{n} = \mathbf{m}$. There are two cases. If $n < m$, then $\vdash n \in m$ so that $n \in n$. Hence, we obtain $\emptyset$. Similarly for the case $m < n$. ∎

For the proof of the next proposition, we recall that the multiplicative conjunction $\otimes$ distributes over additive disjunction $\oplus$:

$$\dfrac{\dfrac{A, C \vdash A \otimes C}{A, C \vdash (A \otimes C) \oplus (B \otimes C)} \quad \dfrac{B, C \vdash B \otimes C}{B, C \vdash (A \otimes C) \oplus (B \otimes C)}}{(A \oplus B) \otimes C \vdash (A \otimes C) \oplus (B \otimes C)} \qquad \dfrac{\dfrac{A, C \vdash (A \oplus B) \otimes C}{A \otimes C \vdash (A \oplus B) \otimes C} \quad \dfrac{B, C \vdash (A \oplus B) \otimes C}{B \otimes C \vdash (A \oplus B) \otimes C}}{(A \otimes C) \oplus (B \otimes C) \vdash (A \oplus B) \otimes C}$$

This allows to combine cases by the multiplicative conjunction and makes the following inference admissible:

$$\dfrac{\Gamma \vdash A_1 \oplus \ldots \oplus A_m \quad \Delta \vdash B_1 \oplus \ldots \oplus B_n}{\Gamma, \Delta \vdash (A_1 \otimes B_1) \oplus \ldots \oplus (A_m \otimes Bn)}$$

where the number of cases on the right is m by n.

Proposition 4. *The following sequents are provable for any* m *and* n*:*

1. $!_2\, sc(x) = sc(\mathbf{m}) \vdash x = \mathbf{m}$,

2. $!_n\, x \in \mathbf{m} \vdash\ !_n\, x = \mathbf{0} \oplus \ldots \oplus\ !_n\, x = \mathbf{m} - \mathbf{1}$.

Proof

1. Since $\vdash x \in sc(x)$, we can derive $sc(x) = sc(\mathbf{m}) \vdash x \in sc(\mathbf{m})$. Furthermore, $x \in sc(\mathbf{m})$ implies $x = \mathbf{0} \oplus \ldots \oplus x = \mathbf{m}$. Then, we have

$$!_2\, sc(x) = sc(\mathbf{m}) \vdash (x = \mathbf{0} \otimes sc(x) = sc(\mathbf{m})) \oplus \ldots (x = \mathbf{m} - \mathbf{1} \otimes sc(x) = sc(\mathbf{m})) \oplus (x = \mathbf{m})$$

and

$$!_2\, sc(x) = sc(\mathbf{m}) \vdash \mathbf{1} = sc(\mathbf{m}) \oplus \ldots (\mathbf{m} = sc(\mathbf{m})) \oplus (x = \mathbf{m}).$$

For $n \leq m$, we can derive $\mathbf{n} = sc(\mathbf{m}) \vdash \emptyset$ since $n \neq m + 1$. Hence,

$$!_2\, sc(x) = sc(\mathbf{m}) \vdash x = \mathbf{m}$$

2. The proof is by induction on n. When $n = 0$, this holds obviously since $x \in \mathbf{m}$ implies $x = \mathbf{0} \oplus \ldots \oplus x = \mathbf{m-1}$. For $n = p+1$, let $\alpha(\mathbf{r})$ be $x = \mathbf{r}$. Then,

$$!_p\, x \in \mathbf{m} \otimes x \in \mathbf{m} \vdash (\, !_p\, \alpha(\mathbf{0}) \oplus \ldots \, !_p\, \alpha(\mathbf{m-1})) \otimes (\alpha(\mathbf{0}) \oplus \alpha(\mathbf{m-1}))$$

We distribute $\otimes$ over $\oplus$. However, if $r \neq r'$, then $\alpha(\mathbf{r}) \otimes \alpha(\mathbf{r'}) \vdash \emptyset$. Hence,

$$!_{p+1}\, x \in \mathbf{m} \vdash \, !_{p+1}\, \alpha(\mathbf{0}) \oplus \, !_{p+1}\, \alpha(\mathbf{m-1})$$

∎

Basic notions We review the definitions of total recursive functions and extend the notion of numeralwise representation in a formal system to linear set theory. We often use the notation $\vec{p}$ for the ordered tuple $\langle p_1, \ldots, p_m \rangle$ for some positive m.

First, let's recall that the class of total recursive functions is the smallest class of numerical functions which includes the following 1–3 and is closed under 4–6.

1. *Zero*: the unary function with the constant value 0;
2. *Successo*: the unary function $p \mapsto p+1$
3. *Projection* for each positive m and n with $n \leq m$: the m-ary function $\vec{p} \mapsto p_n$;
4. *Composition*: $f(\vec{p}) = h(g_1(\vec{p}), \ldots, g_n(\vec{p}))$;
5. *Primitive recursion*:
$$\begin{cases} f(\vec{p}, 0) &= g(\vec{p}), \\ f(\vec{p}, i+1) &= h(f(\vec{p}, i), \vec{p}, i); \end{cases}$$
6. *Minimization* for regular functions: The functions g are called *regular* if for every $\vec{p}$, there exists q such that $g(\vec{p}, q) = 0$. For such regular functions g, we have
$$f(\vec{p}) = \textit{the least number } q \textit{ such that } g(\vec{p}, q) = 0.$$

Secondly, the n-ary numerical functions are *numeralwise representable in a classical formal system* T if there exists an $n+1$-ary predicate F in the language of T such that

- For every $\vec{p}$, if $f(\vec{p}) = q$, then $\begin{cases} \vdash_T F(\vec{\mathbf{p}}, \mathbf{q}), \\ \vdash_T \forall x (F(\vec{\mathbf{p}}, x) \rightarrow x = \mathbf{q}). \end{cases}$

We adopt this definition to linear logic by replacing the classical implication $\rightarrow$ with the linear implication $\multimap$. Furthermore, since the unrestricted comprehension is available, we use the terms in place of predicates. Hence, we say that the n-ary numerical functions are *numeralwise representable in a linear set theory* T if there exists a term F in the language of T such that

- For every $\vec{p}$, if $f(\vec{p}) = q$, then $\begin{cases} \vdash_T \langle \vec{\mathbf{p}}, \mathbf{q} \rangle \in F, \\ \vdash_T \forall x (\langle \vec{\mathbf{p}}, x \rangle \in F \multimap x = \mathbf{q}). \end{cases}$

Zero function Define the term $zero \equiv \{x : \exists y (x = \langle y, 0 \rangle\}$. Then $zero$ numeralwise represents the zero function.

Successor function Define the term $suc \equiv \{x : \exists y(x = \langle y, sc(y)\rangle)\}$. Then suc numeralwise represents the successor function.

Projections Define the term $proj^m_n \equiv \{x : \exists\vec{y}(x = \langle\vec{y}, y_n\rangle)\}$. Then $proj^m_n$ numeralwise the n-the projection function for m-ary tuples.

Composition Suppose that the terms G_i $(1 \le i \le n)$ and H numeralwise represent the m-ary functions g_i and n-ary function h, respectively. Let $comp(\vec{G}, H)$ be the term

$$comp(\vec{G}, H) \equiv \{x : \exists\vec{y}\exists\vec{z}\exists w(x = \langle\vec{y}, w\rangle \otimes \langle\vec{y}, z_1\rangle \in G_1 \otimes \ldots \otimes \langle\vec{y}, z_n\rangle \in G_n \otimes \langle\vec{z}, w\rangle \in H)\}.$$

Then the term numeralwise represents the composition of $\vec{g}$ and h.

Primitive recursion Suppose that the terms G and H numeralwise represent the m-ary function g and $m+2$-ary function h, respectively. Let $A(x, Y)$ be the following predicate

$$\exists\vec{y}\exists z\exists w(x = \langle\vec{y}, z, w\rangle \otimes [(z = \mathbf{0} \otimes \langle\vec{y}, w\rangle \in G) \oplus \exists u\exists v(z = sc(u) \otimes \langle\vec{y}, u, v\rangle \in Y \otimes \langle v, \vec{y}, u, w\rangle \in H)]$$

and define the term $prec(G, H)$ to be the fixpoint of $A(x, Y)$:

$$\vdash (x \in prec(G, H) \multimap A(x, prec(G, H)) \otimes (A(x, prec(G, H)) \multimap x \in prec(G, H)).$$

Then $prec(G, H)$ numeralwise represents the function defined by the primitive recursion from g and h.

Minimization For minimization, we follow the same strategy as the proof of numeralwise representability of total recursive functions in Robinson's Q. Let N^* be the fixpoint

$$x \in N^* \multimapboth x = \mathbf{0} \oplus \exists y(y \in N^* \otimes x = sc(y))$$

and define the term $less$ to be

$$less \equiv \{x : \exists y\exists z(x = \langle y, z\rangle \otimes \exists w(w \in N^* \otimes \langle sc(w), y, z\rangle \in plus\}$$

where $plus$ is the term representing addition.

Proposition 5. *For any natural numbers p and q,*

1. $\vdash \mathbf{p} \in N^*$,
2. *if* $p < q$, *then* $\vdash \langle\mathbf{p}, \mathbf{q}\rangle \in less$.

Proposition 6. *For any natural number i,*

$$\langle sc(x), \mathbf{i}, y\rangle \in plus \vdash \langle x, \mathbf{i+1}, y\rangle \in plus.$$

Proposition 7. *For any natural number i,*

1. $\langle\mathbf{i}, x\rangle \in less \vdash \mathbf{i+1} = x \oplus \langle\mathbf{i+1}, x\rangle \in less$,

2. $\mathbf{i} = x \vdash \langle x, \mathbf{i}+\mathbf{1}\rangle \in less$*,*

3. *(a)* $\langle x, \mathbf{0}\rangle \in less \vdash \emptyset$*,*

(b) if $i \neq 0$*, then* $\langle x, \mathbf{i}\rangle \in less \vdash x = \mathbf{0} \oplus \dots x = \mathbf{i}-\mathbf{1}$*,*

4. $\langle x, \mathbf{i}\rangle \in less \vdash \langle x, \mathbf{i}+\mathbf{1}\rangle \in less$*.*

Lemma 8. *For any natural number* i*,*

$$x \in N^* \vdash \langle x, \mathbf{i}\rangle \in less \oplus x = \mathbf{i} \oplus \langle \mathbf{i}, x\rangle \in less.$$

All the above claims are established by the external induction on i.

Now suppose that the function g is regular and represented by the term G. Define the term $min(G)$ to be

$$min(G) \equiv \{x : \exists \vec{y} \exists z(x = \langle \vec{y}, z\rangle \otimes \langle \vec{y}, z, \mathbf{0}\rangle \in G \otimes z \in N^* \otimes \forall u(\langle u, z\rangle \in less \multimap \langle \vec{y}, u, \mathbf{0}\rangle \notin G)\}.$$

Then the term $min(G)$ numeralwise represents the function f obtained by the minimization from the regular function g.

Hence, we have the following theorem.

Theorem 9. *All total recursive functions are representable in any linear set theory with unrestricted comprehension, equality and pairing.*

5 Some Remarks on the extensionality

As we have already noted, the extensionality principle makes the naive set theory inconsistent even within the contraction-free logics. White consider the infinitary additive extension of affine logic and formulate a naive set theory with the built-in fixpoints based on it. The system is shown to be consistent, but it is also proved that the weak version of extensionality

$$\frac{x \in s \vdash x \in t \quad x \in t \vdash s \in x}{\vdash s = t}$$

makes the system inconsistent[11]. Since we now know that the built-in fixpoints are unnecessary, we have the following corollary of our fixpoint theorem.

Corollary 10. *In affine logic with infinitary additives, the systems with*

1. *the principle of unrestricted comprehension*
2. *the principle of weak extensionality*

are inconsistent.

Note that we do not yet know if such a system is inconsistent or not in the affine logic itself.

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