A sequent calculus for compact closed categories

Masaru Shirahata*
Department of Mathematics
Keio University, Hiyoshi Campus
sirahata@math.hc.keio.ac.jp

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Abstract

In this paper, we introduce the system CMLL of sequent calculus and establish its correspondence with compact closed categories. CMLL is equivalent in provability to the system MLL of classical linear logic with the tensor and par connectives identified. We show that the system allows a fairly simple cut-elimination, and the proofs in the system have a natural interpretation in compact closed categories. However, the soundness of the cut-elimination procedure in terms of the categorical interpretation is by no means evident. We answer to this question affirmatively and establish the soundness by using the coherence result on compact closed categories by Kelly and Laplaza.

1 Introduction

In this paper, we introduce the system CMLL of sequent calculus and establish its correspondence with compact closed categories. CMLL is equivalent in provability to the system MLL of classical linear logic with the tensor $\otimes$ and par $\parr$ connectives identified.

Compact closed categories are abundant in mathematics, for example the finite dimensional vector spaces over a field I, and they were of particular interest to category theorists since they posed a serious obstacle to the general (abstract) treatment of the coherence problem [4, 5].

In computer science, compact closed categories have appeared as a framework for concurrency. The interaction category by Abramsky [1] is such an example, and the connection of Milner's action calculus with reflection [7, 8] to compact closed categories has been investigated.

In linear logic, compact closed categories have been considered at least by Barr [2] and Blute [3]. However, Barr's treatment was purely semantic and

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Blute's syntactic treatment did not fully exploit the structure of compact closed categories.

To be precise, our system of sequent calculus is formulated on top of an arbitrary category $\mathcal{A}$, thus properly called CMLL($\mathcal{A}$). We show that it allows a fairly simple cut-elimination, leaving loops formed by only atomic formulas. We then interpret the system in the free compact closed category $F\mathcal{A}$ on $\mathcal{A}$.

The categorical meaning of our cut-elimination procedure is by no means evident, and one can reasonably suspect that the interpretation (morphism) of a proof is not preserved under the cut-elimination. Therefore, our main interest in this paper is to establish that the cut-elimination procedure indeed preserves morphisms, *i.e.*, its soundness with respect to the categorical interpretation.

We prove the soundness by using the coherence result on compact closed categories by Kelly and Laplaza [6]. They showed that the free compact closed category $F\mathcal{A}$ on $\mathcal{A}$ is isomorphic to the category $G\mathcal{A}$ with the explicit description, the morphisms of which are very much reminiscent of cut-free proof nets for classical linear logic. The soundness of the cut-elimination is established by reinterpreting proofs in $G\mathcal{A}$ and compute the morphisms.

2 Compact closed categories

Intuitively, compact closed categories are symmetric monoidal closed categories in which the closure operation $[A, C]$ is given by $C \otimes A^*$. In the $*$–autonomous categories, $[A, C]$ is given by $(A \otimes C^*)^*$ so that a $*$–autonomous category is compact closed exactly when $(A \otimes B)^* \cong B^* \otimes A^*$.

Definition 1 *In a monoidal category, a left adjoint of an object A is an object A^* with maps*

$$\begin{cases} d_A : I \to A \otimes A^* & \textit{(unit)} \\ e_A : A^* \otimes A \to I & \textit{(counit)} \end{cases}$$

such that the compositions

$$\begin{cases} A \cong I \otimes A \stackrel{d_A \otimes 1}{\to} (A \otimes A^*) \otimes A \cong A \otimes (A^* \otimes A) \stackrel{1 \otimes e_A}{\to} A \otimes I \cong A \\ A^* \cong A^* \otimes I \stackrel{1 \otimes d_A}{\to} A^* \otimes (A \otimes A^*) \cong (A^* \otimes A) \otimes A^* \stackrel{e_A \otimes 1}{\to} I \otimes A^* \cong A^* \end{cases}$$

are the identities, where the isomprphisms are the canonical ones of the monoidal category.

Left adjoints A^* for A may be defined only up to isomorphisms. We choose one of them and call it an *assigned left adjoint*.

Definition 2 *A compact closed category is a symmetric monoidal category in which every object has an assigned left adjoint.*

As we have already noted, a compact closed category is closed with $[A, C] = C \otimes A^*$. On the other hand, a monoidal closed category is compact if and only if the transpose $\hat{f} : A \otimes \to [A, A \otimes I]$ of the map

$$f : (A \otimes [A, I]) \otimes A \cong A \otimes ([A, I] \otimes A) \stackrel{1 \otimes eval}{\to} A \otimes I$$

is an isomorphism for all objects A.

For any category $\mathcal{A}$, we can construct a free compact closed category $F\mathcal{A}$ with the functor $\Phi A : A \to F\mathcal{A}$ in the sense that any functor from $\mathcal{A}$ to a compact closed category $\mathcal{B}$ is uniquely factorized as the composition of ΦA and a functor for compact closed categories (a morphism in the category **Comp** of small compact closed categoies).

The objects of $F\mathcal{A}$ are freely generated from the objects of $\mathcal{A}$ and the formal constant I by the formal binary operator $\otimes$ and the unary operator $(\ )^*$. For morphisms, we first freely generate "arrows" from the morphisms of $\mathcal{A}$, the formal instances of the components of canonical isomorphisms a, r and c of symmetric monoidal categories, and the formal instances of the unit and counit maps. Then, we impose the conditions for canonical maps, units and counits by defining a certain equivalence relation on the arrows and taking the quotient with respect to it. The functor $\Phi\mathcal{A}$ is the obvious embedding. For the precise definition, we refer the reader to Kelly [4, 5], and Kelly and Laplaza [6].

3 The system CMLL of sequent calculus

The system CMLL($\mathcal{A}$) (Compact multiplicative linear logic) is formulated on top of a given category $\mathcal{A}$. The formulas and proof of CMLL are naturally interpreted as objects and morphisms of $F\mathcal{A}$, respectively. Furthermore, a very simple cut-elimination procedure can be given to CMLL.

Definition 3 *The formulas of CMLLL($\mathcal{A}$) are defined inductively:*

1. *For every object A of the category $\mathcal{A}$, we have the atomic formula A;*
2. *The constant I is a formula;*
3. *If X and Y are formulas, so is $X \otimes Y$;*
4. *If X is a formula, so is X^*.*

Axioms

$$(\mathcal{A}) \quad A \vdash_f B \quad \text{for every } f : A \to B \text{ in } \mathcal{A} \qquad\qquad (I) \quad \vdash I$$

Rules of inference

$$(\otimes L) \quad \frac{\Gamma, A, B, , \Delta \vdash \Sigma}{\Gamma, A \otimes B \Delta \vdash \Sigma} \qquad\qquad (\otimes R) \quad \frac{\Gamma \vdash \Delta, A, B, \Sigma}{\Gamma \vdash \Delta, A \otimes B, \Sigma}$$

$$(*L) \quad \frac{\Gamma \vdash A, \Delta}{A^*, \Gamma \vdash \Delta} \qquad\qquad (*R) \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^*}$$

$$(I\ weakening) \quad \frac{\Gamma \vdash \Delta}{I, \Gamma \vdash \Delta}$$

$$(exchange\ L)\quad \frac{\Gamma, A, B, \Delta \vdash \Sigma}{\Gamma, B, A, \Delta \vdash \Sigma} \qquad\qquad (exchange\ R)\quad \frac{\Gamma \vdash \Delta, A, B, \Sigma}{\Gamma \vdash \Delta, B, A, \Sigma}$$

$$(mix)\quad \frac{\Gamma \vdash \Delta \quad \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

$$(cut)\quad \frac{\Gamma \vdash A, \Delta \quad \Sigma, A \vdash \Pi}{\Sigma, \Gamma \vdash \Pi, \Delta}$$

The subscript f in the axioms $(\mathcal{A})$ is to facilitate the categorical interpretation and it can be removed if preferred. Note that the inference rule $(\otimes R)$ comes from the rule for $\invamp$ in MLL.

The cut-elimination of CMLL can be carried out until we only have cuts which are *loops*

$$\dfrac{\dfrac{\dfrac{A \vdash_f A}{\vdash A, A^*}}{\vdash A \otimes A^*} \qquad \dfrac{\dfrac{\dfrac{A \vdash_g A}{A^*, A \vdash}}{A, A^* \vdash}}{A \otimes A^* \vdash}}{\vdash}$$

where the initial sequents are both axioms $(\mathcal{A})$. For this, we first assign the labels to each occurrence of atomic formulas in axioms $(\mathcal{A})$ in a given proof π and extend it to all the occurrences of atomic formulas as subformulas in π in the obvious way. By means of this, we can keep track of which atomic formulas are identified in a cut.

Proposition 4 *The applications of (mix) can be moved upward by the permutations with other rules of inference.*

Proof
By observation. We use $(exchange\ L)$ and $(exchange\ R)$ when necessary. ∎

Definition 5 *A proof π is normal if π contains cuts only as loops.*

The proofs of the form

$$\dfrac{\dfrac{\dfrac{\vdots}{A \vdash A}}{\overline{\overline{A, A^* \vdash}}}}{A \otimes A^* \vdash}$$

will be called the *identity links*, where all the axioms have the form $A_i \vdash_1 A_i$ and the two corresponding occurrences of atomic formulas in A and A^* come from the same axiom.

Theorem 6 *For any proof π, there exists a normal proof $\hat{\pi}$ of the same sequent.*

Proof
We carry out the cut-elimination in four steps. In order to facilitate the soundness proof, we will leave certain redundancy in the procedure.

Step 1 We change cuts into the combination of (mix), $(*R)$, $(\otimes R)$ and the cut with the identity link as follows:

$$\frac{\Gamma \vdash A, \Delta \quad \Sigma, A \vdash \Pi}{\Sigma, \Gamma \vdash \Pi, \Delta} \qquad \mapsto \qquad \frac{\dfrac{\dfrac{\dfrac{\Sigma, A \vdash \Pi \quad \Gamma \vdash A, \Delta}{\Sigma, A, \Gamma \vdash \Pi, A, \Delta}}{\Sigma, \Gamma \vdash \Pi, \Delta, A, A^*}}{\Sigma, \Gamma \vdash \Pi, \Delta, A \otimes A^*} \quad \dfrac{\vdots}{A \otimes A^* \vdash}}{\Sigma, \Gamma \vdash \Pi, \Delta}$$

The original cuts determine the pairing between the labels of the occurrences of atomic formulas in the cut formulas A. We retain the pairing information.

Step 2 We ignore the cuts with the identity link for a while, and delete all applications of inference rules used in forming cut-formulas A and move the applications of (mix) upward. Note that we have extra occurrences of atomic formulas and the constant I originated from the cut formulas.

Step 3 We first remove the extra occurrences of I by simply eliminating the corresponding axioms (I). We then remove the extra atomic formulas by the new cuts with the identity links for them as follows:

$$\frac{\dfrac{\dfrac{\dfrac{\vdots}{\Theta_1, A_i \vdash \Theta_2, A_j}}{\Theta_1 \vdash \Theta_2, A_j, A_i^*}}{\Theta_1 \vdash \Theta_2, A_j \otimes A_i^*} \quad \dfrac{\dfrac{A \vdash_1 A}{A, A^* \vdash}}{A \otimes A^* \vdash}}{\Theta_1 \vdash \Theta_2}$$

where A_i and A_j are the occurrences of the formula A with the labels i and j paired by the original cuts.

Step 4 Finally, we eliminate the cuts with the identity links introduced in the previous step as much as possible. If A_i and A_j in the previous proof figure come from different axioms, we simply carry out the standard cut-elimination procedure for the identity axioms and replace the cut of $B \vdash_f A$ and $A \vdash_g C$ by the new axiom $B \vdash_{g \circ f} C$, retaining the labels for B and C. By this procedure, some of the matching pair of labels may become the labels of the formulas in the same axiom. In that case, we do not have to eliminate the cut with the identity link since we have a loop. The process reduces the number of cut one by one so that it will eventually stop.

∎

We now give the interpretation of proof of CMLL($\mathcal{A}$) in the free compact closed category $F\mathcal{A}$ in $\mathcal{A}$. Let Γ be the list $< A_1, \dots A_n >$. We write $\Gamma^\dagger$ for the object $((\dots(A_1 \otimes A_2)\dots) \otimes A_n)$ in $F\mathcal{A}$. The interpretation of a proof π of the sequent $\Gamma \vdash \Delta$ is denoted $[\![\pi]\!] : \Gamma^\dagger \to \Delta^\dagger$. The morphisms constructed from the identities and the components of a, r, and c by the tensor product and composition will be called *central*.

The interpretation is assigned inductively on the construction of proofs. For the axioms, we have $[\![A \vdash_f B]\!] = f$ and $[\![\vdash I]\!] = 1_I$. The inference rules $(\otimes L)$ and $(\otimes R)$ change only the association of the tensor products by the composition with the appropiate central morphisms. $(Iweakening)$, $(exchangeL)$ and $(exchangeR)$ are handled similarly by the composition with the central morphisms, using r and c in addition to a.

Let π be obtained from τ by $(*L)$. Then $[\![\pi]\!]$ is the morphism:

$$(A^*, \Gamma)^\dagger \cong A^* \otimes \Gamma^\dagger \stackrel{1\otimes[\![\tau]\!]}{\to} A^* \otimes (A, \Delta)^\dagger \cong (A^* \otimes A) \otimes \Delta^\dagger \stackrel{e_A \otimes 1}{\to} I \otimes \Delta^\dagger \cong \Delta^\dagger$$

Similarly, if π is obtained by $(*R)$, we have the morphism:

$$\Gamma^\dagger \cong \Gamma^\dagger \otimes I \stackrel{1\otimes d_A}{\to} \Gamma^\dagger \otimes (A \otimes A^*) \cong (\Gamma^\dagger \otimes A) \otimes A^* \stackrel{[\![\tau]\!]\otimes 1}{\to} \Delta^\dagger A^*$$

If π is obtained from τ_1 and τ_2 by (cut), the interpretation $[\![\pi]\!]$ is the composition:

$$(\Sigma, \Gamma)^\dagger \cong \Sigma^\dagger \otimes \Gamma^\dagger \stackrel{1\otimes[\![\tau_1]\!]}{\to} \Sigma^\dagger \otimes (A, \Delta)^\dagger \cong (\Sigma^\dagger \otimes A) \otimes \Delta^\dagger \stackrel{[\![\tau_2]\!]\otimes 1}{\to} \Pi^\dagger \otimes \Delta^\dagger \cong (\Pi, \Delta)^\dagger$$

If π is obtained by (mix), the morphism is simply the tensor product of $[\![\tau_1]\!]$ and $[\![\tau_2]\!]$:

$$(\Gamma, \Sigma)^\dagger \cong \Gamma^\dagger \otimes \Sigma^\dagger \stackrel{[\![\tau_1]\!]\otimes[\![\tau_2]\!]}{\to} \Delta^\dagger \otimes \Pi^\dagger \cong (\Delta, \Pi)^\dagger$$

4 The category $G\mathcal{A}$

Kelly and Laplaza gave the complete characterization of $F\mathcal{A}$ by showing that $F\mathcal{A}$ is isomorphic to the category $G\mathcal{A}$ with the explicit description of objects and morphisms. We use this result to prove the soundness of the cut-elimination procedure.

We first need to define *cycles* in a given category $\mathcal{A}$. For a category $\mathcal{A}$, the disjoint union $E(\mathcal{A}) = \Sigma_{A \in Ob\mathcal{A}} \mathcal{A}(A, A)$ will be called the *set of endomorphism*.

Definition 7 *The set of cycles $[\mathcal{A}]$ is the quotient set of $E(\mathcal{A})$ modulo the equivalence relation generated by $g \circ f \sim f \circ g$ for $f : A \to B$ and $g : B \to A$.*

Intuitively, a cycle is obtained as follows. A chain of morphisms $A_1 \to A_2 \to \dots \to A_n \to A_1$ need not be distinguished from another chain $A_i \to A_{i+1} \to \dots \to A_1 \to \dots \to A_n \to \dots \to A_{i-1} \to A_i$ composed of the same set of morphisms. The cycle is obtained by identifying those chains which differ only in the starting points.

The morphims of $G\mathcal{A}$ are similar to Kelly-MacLane graphs. However, they may be *incompatible* and some loops may be created by the composition. In order to retain the information on such loops, we eventually associate the interpretation in $\mathcal{A}$ (the functor to $\mathcal{A}$) with the graphs. For the time being, we concentrate on the graph part of morphisms.

Definition 8 *A signed set P is a set $|P|$ together with a function from $|P|$ to $\{-,+\}$.*

We write

1. P^* for the signed set obtained from the signed set P by reversing the signs,
2. I for the empty signed set,
3. 1 for the signed set of one element with the sign $+$,
4. $P \otimes Q$ for the disjoint union of the signed sets P and Q.

Definition 9 *An involution θ is a category which is a coproduct of copies of the category* **2**.

Let $|P|$ be the set of objects in the involution θ. We can then give the signs to the elements of $|P|$ by assigning $-$ if the element is a domain of a non-identitity arrow and $+$ if it is a codomain of a non-identity arrow. We thus obtain a signed set P and the involution θ is called an *involution on P*.

Definition 10 *A loop L is the free category on a graph*

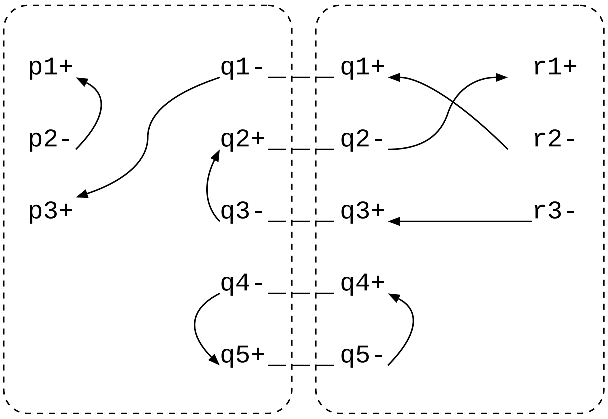
$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_1$$

for $n \geq 1$.

We write $< L >$ for the cycle determined by the composite $f_n \circ \dots \circ f_1$.

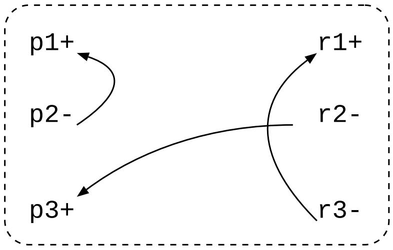
Involutions are used to define the morphisms of $G\mathcal{A}$. Therefore, we need the construction which corresponds to the composition of two morphisms. The coproduct of categories $\mathcal{A}$ and $\mathcal{B}$ will be denoted $\mathcal{A}+\mathcal{B}$. Let θ and ϕ be involutions on $P^* \otimes Q$ and $Q^* \otimes R$, respectively. We then write $\theta +_{|Q|} \phi$ for the pushout of θ and ϕ with respect to the discrete category $|Q|$, which is obtained from $\theta + \phi$ by identifying two copies of $|Q|$.

The operation can be represented graphically. For example,

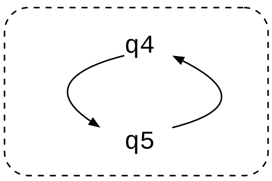


where $|P| = \{p_1, p_2, p_3\}$, $|Q| = \{q_1, q_2, q_3, q_4, q_5\}$ and $|R| = \{r_1, r_2, r_3\}$ with the signs as indicated, the two boxes with dotted lines represent the involutions θ and ϕ, respectively, and $\theta +_{|Q|} \phi$ is obtained by identifying the two occurrences of q_i in the left and right boxes.

The category $\theta +_{|Q|} \phi$ is not an involution on a signed set. However, the subcategory of $\theta +_{|Q|} \phi$ determined by the object set $|P| + |Q|$ is again an involution on the signed set $P^* \otimes Q$. Such a subcategory is denoted $\phi\theta$. For the above example, the involution $\phi\theta$ is represented as follows:



In forming $\theta +_{|Q|} \phi$, we may in fact create loops which do not appear in $\phi\theta$. The information on loops, however, needs to be retained. We write $\phi * \theta$ for the coproduct ΣL_i where L_i's are loops created in forming $\theta +_{|Q|} \phi$ with objects in $|Q|$. In the previous example, $\phi * \theta$ is the loop:



We are now in the position to be able to define the category $G\mathcal{A}$. The objects of $G\mathcal{A}$ are those of $F\mathcal{A}$. To each object X of $G\mathcal{A}$, however, we assign a signed set $P(X)$ and a function $\alpha_X : |P(X)| \to Ob\mathcal{A}$. First, $P(X)$ is assigned inductively as follows:

1. $P(A) = 1$ for the objects A of $\mathcal{A}$;
2. $P(I) = I$;
3. $P(X \otimes Y) = P(X) \otimes P(Y)$;
4. $P(X^*) = (P(X))^*$.

Secondly, the function α_X is also given inductively:

1. $\alpha_A(\bullet) = A$ for the objects A of $\mathcal{A}$ where $\bullet$ is the unique element of 1;
2. $\alpha_I = \emptyset$;
3. $\alpha_{X \otimes Y} = (\alpha_X, \alpha_Y)$, *i.e.* the function defined by cases;
4. $\alpha_{X^*} = \alpha_X$.

For morphisms, we need to take into account cycless which form a monoid. We denote by MV the free commutative monoid on V for any set V.

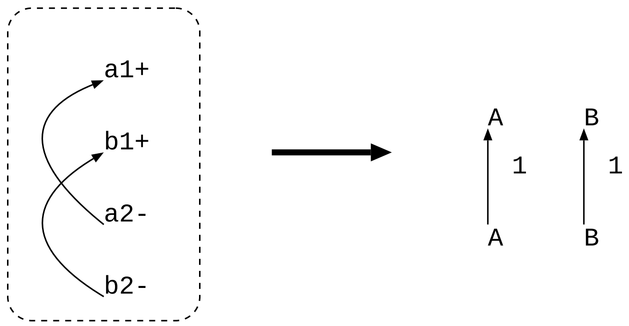
Definition 11 *A morphism $X \to Y$ in $G\mathcal{A}$ is defined to be a triple (θ, p, λ) where*

1. *θ is an involution on $(P(X))^* \otimes P(Y)$;*
2. *p is a functor from θ to $\mathcal{A}$, the object part of which is given by $\alpha_{X^* \otimes Y}$;*
3. *λ is a member of the monoid $M[\mathcal{A}]$ of the cycles.*

The composition of the two morphisms (θ, p, λ) and (ϕ, q, μ) is defined to be $(\phi\theta, s, \lambda + \mu + \Sigma[t] < L_i >)$ where s and t are the restrictions of the functor $(p, q) : \theta +_{|P(Y)|} \phi \to \mathcal{A}$ to $\phi\theta$ and $\phi * \theta$, respectively, and $[t]$ is the function from $[\phi + \theta]$ to $[\mathcal{A}]$ uniquely determined by t. The identity map 1_X is given by the obvious involution on $(P(X))^* \otimes P(X)$ with the functor mapping every arrow to the identity in $\mathcal{A}$ and the empty category as λ.

We note that the two morphisms $f : X \to Y$ and $g : X' \to Y'$ with different domains and codomains may be the same as the triple (θ, p, λ) provided that $P(X)^* \otimes P(Y) = P(X')^* \otimes P(Y)$. To be precise, the morphisms need to be considered always with the domains and codomains. Henceforth, the triple (θ, p, λ) considered in isolation will be called the *body* of the morphism.

The category $G\mathcal{A}$ becomes a compact closed category with the functor $\otimes$ defined by the formal product $X \otimes Y$ on objects and $(\theta, p, \lambda) \otimes (\phi, q, \mu) = (\theta + \phi, (p, q), \lambda + \mu)$ on morphisms, and with the operation $(\ )^*$ on objects defined by the formal X^*. The components of the natural isomorphisms a, r, c are given by the involutions induced by them with the functors mapping every morphism to the identity and the empty cycle. The units are counits are similarly given with the involution induced by the correspondence of X and X^* in $X \otimes X^*$ (or $X^* \otimes X$). We note that the images of the morphisms of such involutions are always identities in $\mathcal{A}$ so that the composition with those morphisms has virtually no effect in the images. For example, the body of the unit $d_{A \otimes B} : I \to (A \otimes B) \otimes (A \otimes B)^*$ is depicted as follows:



Theorem 12 (Kelly and Laplaza [6]) *The free compact closed category $F\mathcal{A}$ on $\mathcal{A}$ is isomorphic to the compact closed category $G\mathcal{A}$.*

5 The soundness of the cut-elimination

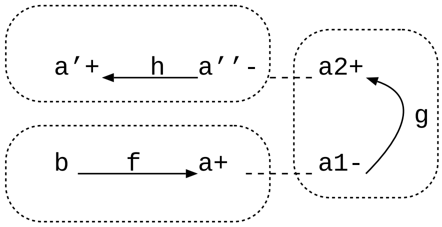
Two involutions are identified if they differ only in the names of objects and arrows. For proving the soundness of our cut-elimination, we simply reinterpret $[\![\pi]\!]$ in $G\mathcal{A}$ and observe that most of the logical operations of CMLL amount to simple renaming and thus preserve the bodies of morphisms.

Proposition 13 *For any central morphism x, the composition $f \circ x$ or $x \circ f$ does not change the body of the morphism f in $G\mathcal{A}$.*

Proof
Let (θ, p, λ) be the body of $f : X \to Y$. The operation $\theta\phi$ with the involution ϕ for x only replaces one copy of the element of $|P(X)|$ by another copy of the same element which has the same image in $\mathcal{A}$. The image of the composition $g \circ h$ in $\phi +_{|P(X)|} \theta$ for g in θ and h in ϕ is the same as the image of g under p since the image of h is the identity. Furthermore, no loop is created. Similarly for $x \circ f$. ∎

Proposition 14 *The operations $(\otimes L)$, $(\otimes R)$, $(*L)$, $(*R)$, $(I$ weakening$)$, $($exchange $L)$ and $($exchange $R)$ do not change the boby of morphisms.*

Proof
The operations except $(*L)$ and $(*R)$ are the compositions with central morphisms. In $(*L)$, we create new nodes (objects) $a' + (-)$, $a'' - (+)$ for X^* and connect $a' + (-)$ with the nodes $a + (-)$ for X through $a'' - (+)$ and $a_1 - (+)$, $a_2 + (-)$ in the involution for e_X, where a, a', a'', a_1 and a_2 are copies of the same element of $|P(X)|$. This is depicted in the following diagram.



In taking the subcategory, we remove the nodes a, a_1, a_2 and a'' which amounts to replace the copy $a+$ by another copy $a'+$. Furthermore, the image of the composition $h \circ g \circ f$ is the same as the original image of f since the images of g and h are the identity. No loop is created in the process. The composition with central morphisms has already been taken care of. Similarly for $(*R)$. ∎

Proposition 15 *The permutation of* (*mix*) *with other rules does not change the body of morphisms.*

Proof
In (*mix*), the bodies of two morphisms are pasted side by side without affecting their internal structures. The effects of the inference rules other than (*cut*) amount to simple renaming which can be carried out after the pasting. For (*cut*), this is immediate by the functoriality of the tensor product. ∎

Lemma 16 *The transformation used in Step 1 of our cut-elimination does not change the interpretation of proofs.*

Proof
Let X be the cut formula and a, a' two copies of $|P(X)|$ to be identified. Suppose that a and a' have the signs $+$ and $-$, respectively, with $f : b- \to a+$ and $g : a'- \to c+$. In the cut, we identify a and a', form the composition $g \circ f$ and remove the intermediate node $a(a')$ to connect b and c directly by $g \circ f$. In our transformation, we add two new copies a_1 and a_2 of the same element as a and a', with the arrow $h : a_1 \to a_2$, and identify a and a' with a_1 and a_2, respectively. After removing the intermediate nodes, we have the arrow $g \circ h \circ f$ directly connecting b and c. However, the image of $g \circ h \circ f$ is identical to the original image of $g \circ f$ since the image of h is the identity. ∎

Theorem 17 *Suppose that $\hat{\pi}$ is obtained from π by our cut-elimination. Then* $[\![\hat{\pi}]\!] = [\![\pi]\!]$, i.e. *our cut-elimination procedure is sound with respect to the categorical interpretation.*

Proof
We have just seen that Step 1 does not change the interpretation. In Step 2, we ignore the cut with identity link and only consider the subproof. As we have already seen, the body of the interpretation of the subproof does not change under Step 2. In Step 3, we simply recover the identifications ignored in Step 2, by creating cuts with the identity links for atomic formulas. Thus, the body

of the interpretation of the proof at Step 3 is the same as the body of the interpretation of the original proof. Step 4 amounts to removing intermediate nodes and does not change the body at all. ∎

6 Concluding remarks

The coherence problem of compact closed categories with biproduct has been studied by Soloviev [9]. We also considered the extensions of CMLL with finite and infinitary additives (product and coproduct). The introduction of infinitary additives was motivated by the fact that the exponential $!A$ can be defined by the infinitary additive formula $1 \& A \& (A \otimes A) \& \ldots$ in compact closed categories, as observed by Barr [2] and used by Abramsky [1]. However, it turned out that there is a counterexample to the cut elimination even with only finite additives and the system becomes inconsistent with infinitary additives in the sense that any sequent is derivable in the system. This seems to pose an interesting question of the importance of provability in linear logic.

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