Distance-Restricted Matching Extension
in Triangulations on the Torus and the Klein Bottle

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Abstract
A graph $G$ with at least $2m + 2$ edges is said to be distance $d$ $m$-extendable if for any matching $M$ of $G$ with $m$ edges in which the edges lie pair-wise distance at least $d$, there exists a perfect matching of $G$ containing $M$. In [J. Graph Theory 67 (2011), no. 1, 38-46], Aldred and Plummer proved that every 5-connected triangulation on the plane or the projective plane with an even order is distance 5 $m$-extendable for any $m$. In this paper we prove that the same thing holds for every triangulation on the torus or the Klein bottle.

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1 Introduction

We consider only simple graphs without loops or multiple edges. A set $M$ of edges in a graph is said to be a matching if no two members of $M$ share a vertex. A perfect matching of a graph $G$ is a matching of $G$ which covers all the vertices of $G$. If a matching $M$ of $G$ is a subset of a perfect matching in $G$, then $M$ is said to be extendable in $G$, and a graph with at least $2m + 2$ vertices in which every matching of size $m$ is extendable is called $m$-extendable.

A closed curve on a closed surface $F^2$ is a continuous function $l : S^1 \rightarrow F^2$ or its image, where $S^1$ is the 1-dimensional sphere, that is, $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. A closed curve $l$ is called simple if the function $l$ is an injection. Moreover, a simple closed curve $l$ is called essential if it does not bound a 2-cell on $F^2$.

In this paper we deal with a graph $G$ embedded on some closed surface, and investigate the property for a matching $M$ in $G$ to be extendable. For a matching with at most two edges, Plummer [10] proved the following.

**Theorem 1.** Every 5-connected planar graph with an even order is 2-extendable.

To show a similar result for any graph $G$ on some other surface $F^2$, Kawarabayashi et al. added a condition on the representativity $\rho(G)$ (the minimum number $r$ such that any essential simple closed curve on $F^2$ meets $G$ in at least $r$ places), and proved the following.

**Theorem 2** ([7]). Every 5-connected graph $G$ with an even order embedded on a closed surface $F^2$, except the sphere, is 2-extendable if $\rho(G) \geq 7 - 2\chi(F^2)$, where $\chi(F^2)$ is the Euler characteristic of $F^2$.

Moreover, they proved that there are infinitely many 5-connected triangulations $G$ on $F^2$ with $\rho(G) = 3$ that are not 2-extendable, for any closed surface $F^2$ except the sphere and the projective plane. Therefore, the condition on the representativity is necessary in Theorem 2 when $F^2$ is not the projective plane.

It is known that no planar graph with at least 8 vertices is 3-extendable ([9]). Later, Aldred and Plummer ([1], [3], [4]) studied the distance between the edges in matchings which are not extendable. A graph $G$ with at least $2m + 2$ edges is said to be distance $d$ $m$-extendable if any matching $M$ of $G$ with $m$ edges in which the edges lie pair-wise distance at least $d$ is extendable. In particular, several important results for proximity based matching extension have been obtained for triangulations, i.e. simple graphs which are embedded on surfaces so that each face is bounded by a triangle.

**Theorem 3** ([1]). Every 5-connected triangulation on the plane with an even order is distance 2 3-extendable.

It follows from Theorem 3 that, in any 5-connected triangulation on the plane with an even order, any non-extendable matching with three edges contains a pair of edges with distance 1.
It is shown in [3] that the conclusion of Theorem 3 cannot be extended to “distance 2 4-extendable”. However, if the pair-wise distance of the given matching is increased, then we can extend 4 or more edges.

**Theorem 4** ([3], [4]). Every 5-connected triangulation on the plane with an even order is distance 3 4-extendable. Moreover, there exist infinitely many 5-connected triangulations on the plane with an even order which are not distance 3 10-extendable.

**Theorem 5** ([4]). Every 5-connected triangulation on the plane with an even order is distance 4 7-extendable.

Note that, in distance two or three case, there exist a maximum value on the number of edges to extend. Though we do not know whether such a maximum exists or not in distance four case, it disappears in distance five case.

**Theorem 6** ([4]). Every 5-connected triangulation on the plane with an even order is distance 5 m-extendable for any m.

It is also shown in [4] that Theorem 6 still holds when “plane” is replaced by “projective plane”. For the graphs on the torus or Klein bottle, it follows from Mizukai et al.’s result [8] that every 5-connected triangulation on the torus or the Klein bottle with an even order is distance 3 2-extendable for any m. This result is sharp in the following sense.

**Proposition 7** (Aldred and Plummer, [5]). There are 5-connected triangulations of both the torus and the Klein bottle with an even order which are not distance 3 3-extendable.

The main purpose of this paper is to extend Theorem 6 to graphs on the torus and the Klein bottle. In the above results, we always find a difference between planar case and the toroidal or the Klein bottle case: when we extend Theorem 1 to Theorem 2, the additional condition on representativity is necessary, and in Proposition 7, the number of the edges to extend is reduced compared to the planar case. In contrast to these situations, we can extend Theorem 6 without any extra-condition.

**Theorem 8.** Let G be a 5-connected triangulation on the torus or the Klein bottle with an even order. If \( m \geq 0 \) and G has at least \( 2m + 2 \) vertices, then G is distance 5 m-extendable.

It is shown in [5] that there exist 4-connected triangulations of both the torus and the Klein bottle with an even order which are not 1-extendable, and so the condition of the connectivity in Theorem 8 cannot be relaxed. Next we show that the condition “triangulation” cannot be dropped in Theorem 8. Let \( G_0 \) and \( H_0 \) be the graphs shown in Figures 1 and 2, respectively. Let G be the graph obtained from \( G_0 - \{w_1, w_2, w_3, w_4\} \) by adding 4 copies \( H_1, H_2, H_3, H_4 \) of \( H_0 - \{v\} \) and joining five vertices of degree 4 in \( H_i \) and the five neighbors of \( w_i \) in \( G_0 \) by a perfect matching for \( i = 1, 2, 3, 4 \) so that the resulting graph is toroidal. Then, since both of \( G_0 \) and \( H_0 \) is 5-connected, G
is also 5-connected. Moreover, since $G - \{x_1, y_1, z_1, x_2, y_2, z_2\}$ has 4 odd components, $\{x_1y_1, x_2y_2\}$ is not extendable in $G$. Thus $G$ is not distance 5 $m$-extendable, though $G$ is a 5-connected toroidal graph with an even order. Note that we obtain larger examples by recursively exchanging a vertex in $V(G) \setminus \{x_1, y_1, z_1, x_2, y_2, z_2\}$ for the graph $H_0 - \{v\}$ and adding 5 edges as indicated in the above. Moreover, notice that $G$ can be embedded on the Klein bottle (such an embedding is obtained by changing the order of the edges between $\{w_3, w_4\}$ and $\{x_1, y_1, z_1\}$ and changing the direction of the right border in Figure 1).

Though Theorem 8 is best possible in the above senses, it remains to be studied whether we can replace “distance 5” with “distance 4” in Theorem 8 or not.

We prove Theorem 8 in the next section. In the rest of this section, we prepare terminology and notation used in the proof. Two closed curves $l_1$ and $l_2$ on a closed surface $F^2$ are said to be homotopic to each other on $F^2$ if there exists a continuous function $\Phi : [0,1] \times S^1 \to F^2$ such that $\Phi(0,x) = l_1(x)$ and $\Phi(1,x) = l_2(x)$ for each $x \in S^1$. A closed curve $l$ on $F^2$ is said to be 1-sided if a tubular neighborhood of $l$ forms a Möbius band, and 2-sided otherwise.

An essential (resp. 1-sided, 2-sided, separating) cycle of a graph on $F^2$ is a cycle whose edges induce an essential (resp. 1-sided, 2-sided, separating) curve. Note that a separating cycle $C$ on $F^2$ separates the surface, but not necessarily the graph. And so the graph $G - V(C)$ may be connected. Let $C$ be a 2-sided cycle of a graph $G$ on $F^2$ and let $D_1, D_2$ be vertex-disjoint subgraphs of $G - V(C)$. If there exist two edges $x_1y_1$ and $x_2y_2$ such that $x_1, x_2 \in V(C)$, $y_i \in V(D_i)$ for $i = 1, 2$ and $x_1y_1$ is adjacent to $x_1$ on the right-hand side of $C$ and $x_2y_2$ is adjacent to $x_2$ on the left-hand side of $C$ for an arbitrary orientation of $C$, then we say that $D_1$ and $D_2$ are adjacent to $C$ on opposite sides of $C$.

Let $e$ be an edge and $F$ be a set of edges. The set of two vertices which are incident with $e$ is denoted by $V(e)$, and we denote $V(F) = \bigcup_{e \in F} V(e)$. Moreover, the length of a shortest path joining a vertex $v$ and $V(e)$ is denoted by $d(v,e)$. For other terminology and notation, we refer the readers to [6].
2 Proof of Theorem 8

Suppose that $m$ is the smallest integer for which distance $5$ $m$-extendable fails to hold. Since every $5$-connected triangulation on the torus (resp. the Klein bottle) with an even order is $1$-extendable by Theorem 4.3 (b) (resp. Theorem 5.3(b)) of [2], we have $m \geq 2$. Let $M = \{e_1, \ldots, e_m\}$ be a set of $m$ edges at mutual distance five or more which does not extend to a perfect matching. Then since $G' = G - V(M)$ does not have a perfect matching, by Tutte’s theorem there is a barrier set $S \subseteq V(G)$ such that $c_o(G' - S) \geq |S| + 2$, where $c_o(H)$ denotes the number of odd components of a graph $H$. By the minimality of $m$ we have $c_o(G' - S) = |S| + 2$. Now we choose such an $S$ to be as small as possible.

Let $K = S \cup V(M)$. Assume that there exists a component $D$ of $G - K$ such that two vertices $r_1, r_2$ of $S$ is contained in different components of $G - V(D)$. Let $H'$ be the component of $G - V(D)$ containing $r_1$, let $R_3 = V(H') \cap S$ and let $R_2 = S \setminus R_1$. Then $c_o(G - R_i) \geq |R_i| + 2$ holds for $i = 1$ or $2$, a contradiction. Therefore,

$$G - V(D) \text{ is connected for every component } D \text{ of } G - K. \quad (1)$$

We construct the bipartite distillation $G^*$ on graph $G$ as in the proof of Theorem 2.1 of [4]. Delete all the even components of $G - K$ and all edges within $G[K]$. Next contract each of the odd components of $G - K$ to a single vertex and delete any loops and multiple edges. We call the resulting bipartite graph $G^*$. Since $G^*$ is a minor of $G$, we can embed $G^*$ on the same closed surface as $G$, the torus or the Klein bottle. So $|E(G^*)| \leq 2|V(G^*)| = 2(2|S| + 2 + 2m) = 4|S| + 4m + 4$. On the other hand, since $G$ is $5$-connected, every odd component of $G - K$ has at least five vertices of attachment in $K$. And so $|E(G^*)| \geq 5(|S| + 2)$. Thus

$$|S| \leq 4m - 6. \quad (2)$$

Suppose a vertex $s \in S$ has neighbors in at most two components of $G' - S$. Let $S' = S - \{s\}$, then $G' - S'$ has at least $|S| + 1 = |S'| + 2$ odd components, which contradicts the minimality of $S$. Therefore each vertex $s \in S$ has neighbors in at least three components of $G' - S$. Take $s \in S$ and let the neighbors of $s$ be $v_1, v_2, \ldots, v_l$ such that $sv_iv_{i+1}$ is a face of $G$ for each $i$ with $1 \leq i \leq l - 1$. Let $D_1, D_2$ and $D_3$ be three odd components of $G' - S$ containing a neighbor of $s$. Without loss of generality, we may assume that $v_1 \in D_1, v_p \in D_2$ and $v_q \in D_3$ with $1 < p < q$. Then, since each $D_i$ is a different odd component of $G - K$ and $G$ is a triangulation, we have three vertices $v_{p'}, v_{q'}, v_{r'} \in K$ such that $1 < p' < p < q' < q < r'$. From this observation, it follows that each vertex $s$ in $S$ has at least three distinct neighbors $t_1, t_2$ and $t_3$ in $K$.

For each $e_i \in M$, let $S(e_i) = \{s \in K \mid 1 \leq d(s, e_i) \leq 2\}$. Since edges in $M$ are at least distance five apart, it holds that $S(e_i) \subseteq S$ and $S(e_i) \cap S(e_j) = \emptyset$ for $e_i, e_j \in M$ with $i \neq j$. The next claim plays a central role in our proof of Theorem 8. Note that $G$ satisfies $ii')$ or $ii'')$ of Claim 1 only when $G$ is embedded in the Klein bottle.
Claim 1. For each edge \( e_i = xy \in M \), at least one of the following holds.

\[ i) \ |S(e_i)| \geq 4. \]

\[ ii) \ |S(e_i)| \geq 2 \text{ and } G[S(e_i) \cup \{x, y\}] \text{ contains two non-homotopic essential cycles of length at most 4.} \]

\[ ii') \ |S(e_i)| \geq 2 \text{ and } G[S(e_i) \cup \{x, y\}] \text{ contains a separating 2-sided essential cycle of length at most 4.} \]

\[ ii'') \ |S(e_i)| \geq 2 \text{ and } G[S(e_i) \cup \{x, y\}] \text{ contains a 1-sided essential cycle of length at most 4.} \]

\[ iii) \ |S(e_i)| \geq 3, G[S(e_i) \cup \{x, y\}] \text{ contains a non-separating 2-sided essential cycle of length at most 4 and there exists a component of } G - K \text{ which has a neighbor of a vertex of } S(e_i) \text{ and contains a non-separating 2-sided essential cycle.} \]

Proof. We assume that neither i), ii), ii') nor ii'') holds, and prove iii).

Note that \( e_i \) has neighbors in at least two odd components of \( G - K \), as otherwise \( c_2(G'' - S) = |S| + 2 \), where \( G'' = G - (V(M) \setminus \{x, y\}) \), contradicting the minimality of \( m \). Now contract \( e_i \) to a single vertex \( w_i \) and delete the created loop. Since \( G \) is a 5-connected triangulation, this contraction creates exactly two pairs of multiple edges which bound a 2-cell (in particular, such pairs do not share an edge, and each pair induces a digon). Here delete one of the edges of each pair, and let \( G_1 \) be the resulting graph. Then \( G_1 \) is a triangulation as well. Let the neighbors of \( w_i \) in \( G_1 \) be \( v_1, v_2, \ldots, v_l \) such that \( w_i,v_jv_{j+1} \) is a face of \( G_1 \) for each \( j \) with \( 1 \leq j \leq l - 1 \). Without loss of generality, we may assume that \( v_1 \) and \( v_p \) belong to different odd components of \( G - K \) for some \( p \). Let \( D \) (resp. \( D' \)) be the component of \( G - K \) which contains \( v_1 \) (resp. \( v_p \)). Then we have two vertices \( v_{p'}, v_{p''} \in S \) such that \( 1 < p' < p < p'' < l \). Let \( s_1 = v_{p'} \) and \( s_2 = v_{p''} \).

Now \( s_1 \) is a neighbor of the edge \( e_i = xy \) in \( G \). Without loss of generality, we may assume that \( s_1x \in E(G) \). As we noted before, \( s_1 \) has neighbors in at least three odd components and has at least three neighbors in \( K \). Let \( D_1, D_2, \ldots, D_k \) be such components (\( k \geq 3 \)), and let \( x, u_1, u_2, \ldots, u_{k-1} \) be such neighbors, where \( k \geq 3 \) and \( x, N(s_1) \cap D_1, u_1, N(s_1) \cap D_2, u_2, N(s_1) \cap D_3, \ldots, u_{k-1}, N(s_1) \cap D_k \) appear in this order when we traverse \( N(s_1) \) in the clockwise direction (see Figure 3).

Subclaim 1. \( G[S(e_i) \cup \{x, y\}] \) contains an essential cycle \( C \) of length at most 4 which contains the edge \( s_1x \). Moreover, if \( C \) is a 2-sided cycle, then there exists two distinct components of \( G - K \) which are adjacent to \( C \) on opposite sides of \( C \).

Proof. In the case \( s_1 = s_2 \), let \( C = xys_1 \). By the construction of \( G_1 \), \( C \) is an essential cycle. Moreover, if \( C \) is a 2-sided cycle, then \( D \) and \( D' \) are adjacent to \( C \) on opposite sides of \( C \). Thus \( C \) is a desired cycle.

In the case \( u_j = y \) for some \( j \), let \( C = s_1u_jx \). If \( C \) is not an essential cycle, then \( D_1 \) and \( D_k \) are contained in different components of \( G - V(C) \), contradicting the 5-connectedness of \( G \). Therefore \( C \)
is an essential cycle. Moreover, if $C$ is a 2-sided cycle, then $D_1$ and $D_k$ are adjacent to $C$ on opposite sides of $C$, and thus $C$ is a desired cycle.

Therefore, we may assume that $s_1 \neq s_2$ and $u_j \neq y$ for each $j$. Since i) does not hold, it follows that $u_r = s_2$ for some $r$. Then either $s_1 y x$ or $s_1 u_r x$ is a cycle, say $C$. If $C$ is not an essential cycle, then $D_1$ and $D_k$ are contained in different components of $G - V(C)$, contradicting the 5-connectedness of $G$. Therefore $C$ is an essential cycle. Moreover, if $C$ is a 2-sided cycle, then $D_1$ and $D_k$ are adjacent to $C$ on opposite sides of $C$, and thus $C$ is a desired cycle.

Take $C$ as in Subclaim 1. If $C$ is a 1-sided cycle or a separating 2-sided cycle, then since $\{s_1, u_1, u_2, \ldots, u_k\} \setminus \{x, y\} \subseteq S(e_i)$, we have $|S(e_i)| \geq 2$. This contradicts the assumption that neither ii') nor ii") holds, and thus we may assume that $C$ is a non-separating 2-sided cycle. Recall that $s_1$ has neighbors in at least three odd components. Without loss of generality, we may assume that $s_1$ is adjacent to $D_1$ and $D_2$ on the left-hand side of $C$ and is adjacent to $D_k$ on the right-hand side of $C$. Let $S'(e_i)$ be the set of the vertices in $S(e_i)$ which are in $V(C)$ or are adjacent to $C$ on the left-hand side. We call a component $D$ of $G - K$ essential if $D$ contains an essential cycle.

**Subclaim 2.** $|S'(e_i)| \geq 3$.

**Proof.** If $|V(C) \cap S| \geq 2$, then $(V(C) \cap S) \cup \{u_1\}$ establishes Subclaim 2. Thus we may assume $|V(C) \cap S| = 1$, which implies $s_1 = s_2$ and $C = xys_1$. Let the neighbors of $s_1$ in $G$ be $a_1, a_2, \ldots, a_l$ such that $s_1 a_j a_{j+1}$ is a face of $G$ for each $j$ with $1 \leq j \leq l - 1$. Without loss of generality, we may assume that $x = a_1$, $u_1 = a_p$, and $y = a_q$. If there exist $j$ with $1 < j < q$ and $j \neq p$ such that $a_j \in S(e_i)$, then $\{s_1, u_1, a_j\}$ implies the assertion. Thus we may assume that

$$a_j \not\in S(e_i) \text{ for every } j \text{ with } 1 < j < q \text{ and } j \neq p.$$ (3)

Since $C$ is a non-separating 2-sided cycle, it follows that either $D_1$ or $D_2$ is not an essential component. Assume that $D_1$ is not an essential component. Let $T_1$ be the set of vertices of $K$ which
are adjacent to $D_1$, then by (1) $T_1$ induces a closed trail. Let $b_1 b_2 b_3 \ldots$ be such a closed trail, where $b_1 = s_1$ and $b_2 = u_1$. Moreover, let $l$ be the smallest number such that $b_l \in V(C)$ and the edge $b_{l-1} b_l$ is adjacent to $C$ on the left-hand side. By (3), we have $b_l = x$ or $y$. Thus $b_{l-1} \in S(e_i) \setminus \{s_1\}$. If $b_{l-1} = u_1$, then $D_1$ and $D_2$ are contained in different components of $G - \{s_1 b_{l-1} b_l\}$, a contradiction. Therefore we have $b_{l-1} \neq u_1$, and hence $\{s_1, u_1, b_{l-1}\}$ implies the assertion. By the same argument, we also obtain the assertion in the case where $D_2$ is not an essential component. \hfill $\Box$

If $D_k$ is an essential component, then since $C$ is a non-separating 2-sided cycle, the essential cycle contained in $D_k$ is also a non-separating 2-sided cycle. Hence iii) follows from Subclaims 1 and 2. Thus we assume that $D_k$ is not an essential component. Let $T_k$ be the set of vertices of $K$ which are adjacent to $D_k$. Since $G$ is 5-connected, we have $|T_k| \geq 5$, which implies $T_k \setminus V(C) \neq \emptyset$. Since $G[T_k]$ is connected, there exists $s' \in T_k \setminus V(C)$ such that $s'$ is adjacent to $C$ on the right-hand side. Then $s' \in S(e_i)$. If $s' \notin S'(e_i)$, then $S'(e_i) \cup \{s'\}$ implies i), a contradiction. On the other hand, if $s' \in S'(e_i)$, then since $s'$ is adjacent to $C$ on both sides, ii) holds. This contradiction completes the proof of Claim 1. \hfill $\Box$

**Claim 2.** If ii') or ii") of Claim 1 holds for some $e_i$, then there exists at most one edge in $M \setminus \{e_i\}$ which satisfies ii') or ii") and no edge in $M \setminus \{e_i\}$ satisfies ii) or iii).

**Proof.** Recall that $G$ is on the Klein bottle when ii') or ii") holds for $e_i$. Assume first that $e_i$ satisfies ii') and let $C$ be the separating essential cycle of length at most 4 contained in $G[S(e_i) \cup \{x, y\}]$. By cutting the Klein bottle along $C$ (and deleting the vertices in $C$), we obtain two Möbius bands. Since $G$ is 5-connected, one of them cannot contain any vertex. Moreover, since $S(e_i) \cap S(e_j) = \emptyset$ for every $j \neq i$, the other Möbius band can contain at most one edge in $M$ which satisfies ii') or ii") and it cannot contain any edge in $M$ which satisfies ii) or iii).

Next assume that no edge in $M$ satisfies ii') and assume that $e_i$ satisfies ii''). Let $C$ be the 1-sided essential cycle of length at most 4 contained in $G[S(e_i) \cup \{x, y\}]$, then by cutting the Klein bottle along $C$, we obtain a Möbius band. Since $S(e_i) \cap S(e_j) = \emptyset$ for every $j \neq i$, the Möbius band can contain at most one edge in $M$ which satisfies ii''), and it cannot contain any edge in $M$ which satisfies ii) or iii). \hfill $\Box$

By Claim 2, if ii') or ii'') of Claim 1 holds for some $e_i$, then we can choose $e_j \neq e_i$ so that each edge in $M \setminus \{e_i, e_j\}$ satisfies i) of Claim 1. Then $|S| \geq \sum_{i=1}^{m} |S(e_i)| \geq 4(m-2) + 2 + 2 \geq 4m - 4$, which contradicts (2). Thus we may assume that no edge in $M$ satisfies ii') or ii'') of Claim 1. Moreover, if ii) of Claim 1 holds for some $e_i$, then since $S(e_i) \cap S(e_j) = \emptyset$ for every $j \neq i$, each edge in $M \setminus \{e_i\}$ satisfies i) of Claim 1. Then $|S| \geq \sum_{i=1}^{m} |S(e_i)| \geq 4(m-1) + 2 \geq 4m - 2$, which contradicts (2). Thus we may assume that no edge in $M$ satisfies ii) of Claim 1.

If there exists no essential component of $G - K$, then iii) of Claim 1 does not hold for any $e_i$. Then $|S| \geq \sum_{i=1}^{m} |S(e_i)| \geq 4m$, which contradicts (2). Hence there exists an essential component of $G - K$. Let $D_1, \ldots, D_k$ be the essential components of $G - K$. In the case $k \geq 3$, let $H_1, \ldots, H_l$ be
the components of $G - (V(D_1) \cup \ldots \cup V(D_k))$. Moreover, let $S_i = S \cap H_i$ and let $c_i$ be the number of components of $G - K$ which is contained in $H_i$. Assume that $c_j \leq |S_j| - 1$ for some $j$. Then $G' - (S - S_j)$ has at least $|S| + 2 - (c_j + 1) \geq |S| + 2 - |S_j| = |S - S_j| + 2$ components, which contradicts the minimality of $S$. Thus $c_i \geq |S_i|$ for every $i$. However, $c_o(G - K) = k + \sum_{i=1}^l c_i \geq k + \sum_{i=1}^l |S_i| \geq |S| + 3$, a contradiction. Therefore we have $k \leq 2$. Since each $D_i$ has two boundary walks which contain an essential cycle, at most four edges of $M$ satisfies iii) of Claim 1. Therefore $|S| \geq \sum_{i=1}^m |S(e_i)| \geq 4(m - 4) + 3 \cdot 4 = 4m - 4$, which contradicts (2). \qed

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