Hamiltonian cycles in bipartite toroidal graphs with a partite set of degree four vertices

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Abstract
Let $G$ be a 3-connected bipartite graph with partite sets $X \cup Y$ which is embeddable in the torus. We shall prove that $G$ has a Hamiltonian cycle if (i) $G$ is balanced, i.e., $|X| = |Y|$, and (ii) each vertex $x \in X$ has degree four. In order to prove the result, we establish a result on orientations of quadrangular torus maps possibly with multiple edges. This result implies that every 4-connected toroidal graph with toughness exactly one is Hamiltonian, and partially solves a well-known Nash-Williams’ conjecture.

1 Introduction
A surface is a connected compact 2-dimensional manifold without boundary. A map on a surface $F^2$ means a fixed embedding of a graph on $F^2$. A Hamiltonian cycle of a graph $G$ is a cycle passing through all vertices of $G$ exactly once. A graph $G$ is said to be Hamiltonian if $G$ has a Hamiltonian cycle. We would like to consider whether graphs on surfaces have Hamiltonian cycles.

Whitney proved that any 4-connected plane triangulation is Hamiltonian [15], and Tutte improved this result for 4-connected plane graphs [14]. Starting from these results, Hamiltonicity of graphs on surfaces has been extensively studied, for example, Thomas and Yu proved that every 4-connected projective plane map is Hamiltonian [10]. Note that in those theorems, the 4-connectedness cannot be omitted, since some non-4-connected graphs on those surfaces have no Hamiltonian cycles. So next we are interested in is the toroidal case. Actually, Nash-Williams posed the following conjecture.

Conjecture 1 (Nash-Williams [9]). Every 4-connected torus map is Hamiltonian.

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This conjecture has attracted much attention for long years, but it is still open. Actually, many researchers gave some partial solutions to Conjecture 1. Barnette [4], and Brunet and Richter [5] showed that every 5-connected toroidal triangulation has a Hamiltonian path, and a Hamiltonian cycle, respectively. Thomas and Yu improved these results and showed that every 5-connected toroidal map is Hamiltonian [11], and later, they showed the existence of Hamiltonian paths in 4-connected toroidal graphs [12]. Note that the proofs of the above results used the concept of a “Tutte path or cycle”. For more informations about it, the readers should refer Section 3 of a nice survey [6]. As mentioned later, we also give a partial solution to Conjecture 1 in this paper, but our method is very different from those using a “Tutte path or cycle”.

A quadrangulation on a surface $F^2$ is a map of a simple graph on $F^2$ such that each face is bounded by a 4-cycle. Altshuler proved that every 4-connected torus quadrangulation has a Hamiltonian cycle [2]. It is easy to see that a torus quadrangulation is 4-connected if and only if it is 4-regular, and there is a simple standard form of 4-regular torus quadrangulations with a rectangular grid, which helps us to find Hamiltonian cycles in them. (Similarly, the Hamiltonicity of 6-regular torus triangulations is verified in the same paper by using a standard form of them [1]. Alspach and Zhang showed that the dual of such graphs is also Hamiltonian using algebraic approach [3].)

In this paper, restricting bipartite quadrangulations but relaxing the 4-connectivity, we shall prove the following.

**Theorem 2.** Let $Q$ be a 3-connected bipartite torus quadrangulation. If one of the two partite sets consists only of degree four vertices, then $Q$ is Hamiltonian.

Let us consider the condition of $Q$ to be a quadrangulation in the above theorem. Let $Q$ be a 3-connected bipartite graph on the torus with partite sets $X$ and $Y$, and assume that every vertex in $X$ has degree 4. Then by Euler’s formula, $Q$ is a quadrangulation if and only if $|E(Q)| = 2|V(Q)|$. On the other hand, since $|E(Q)| = \sum_{x \in X} \deg(x) = 4|X|$, it follows that $Q$ is balanced (i.e., $|X| = |Y|$) if and only if $Q$ is a torus quadrangulation (i.e., $|E(Q)| = 2|V(Q)|$). Hence, exchanging these two conditions in Theorem 2, we obtain Theorem 3, which is equivalent to Theorem 2 but will be more appealing, since the balance of $X$ and $Y$ is a trivial necessary condition for $Q$ to be Hamiltonian.

**Theorem 3.** Let $Q$ be a 3-connected balanced bipartite graph which is embeddable in the torus. If one of the two partite sets consists only of degree four vertices, then $Q$ is Hamiltonian.

In Section 2, we will make some remarks on Theorems 2 and 3, and in Sections 3–5 we will prove Theorem 2.

Here we put definitions we need later. A cycle $C$ of length $k$ is called a $k$-cycle. If $k$ is even (resp., odd), then $C$ is said to be even (resp., odd). A simple closed curve $l$ on a non-spherical surface $F^2$ is said to be essential if $l$ does not bound a 2-cell on $F^2$. An essential cycle of a graph on $F^2$ is a cycle whose edges induce an essential curve. A $k$-vertex (resp., a $k$-face) is a vertex (resp., face) of degree exactly $k$. 

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The representativity of a map $G$ on a surface $F^2$, denoted $r(G)$, is the minimum number of intersecting points of $G$ and $\gamma$, where $\gamma$ ranges over all essential simple closed curves on $F^2$. We say that $G$ is $k$-representative if $G$ has representativity at least $k$.

Let $H$ be a map on a surface $F^2$ and we suppose that all vertices of $H$ are colored by black. The face subdivision of $H$, denoted $\tilde{H}$, is obtained from $H$ by adding a new white vertex to each face of $H$ and joining it to all vertices lying on the corresponding face boundary. The radial graph of $H$, denoted $R(H)$, is obtained from $\tilde{H}$ by removing all edges joining two black vertices. It is easy to see that $R(H)$ is bipartite and each face of $R(H)$ is quadrilateral. Moreover, $R(H)$ is simple if and only if each face of $H$ is bounded by a cycle.

We say that a graph $G$ is $k$-tough if $\frac{|S|}{\omega(G-S)} \geq k$ for any $S \subset V(G)$ with $G - S$ disconnected, where $\omega(\cdot)$ is the number of components. The toughness of $G$ is defined to be the maximum real number $k$ such that $G$ is $k$-tough. It is well-known that if $G$ is Hamiltonian, then $G$ is 1-tough.

## 2 Remarks on Theorems 2 and 3

In this section, we make four remarks on Theorems 2 and 3.

3-Connectedness and a partite set of 4-vertices. Let us consider whether the conditions in Theorem 2 can be omitted. Let $K$ be a 4-regular bipartite torus quadrangulation, which must be balanced with black and white vertices. Let $S$ and $T$ be the two bipartite plane graphs shown in Figure 1 (1) and (2), respectively. Let $T'$ be the graph $T$ with white and black interchanged.

We first consider the 3-connectedness. Let $f$ be a face of $K$. Let $Q$ be the bipartite torus quadrangulation obtained from $K$ and $S$ by pasting the boundary 4-cycles of $f$ and that of $S$ so that vertices with the same color are identified. Then $Q$ is a bipartite torus quadrangulation in which all white vertices are of degree four, but it is not 1-tough, since $Q$ with the two white inner vertices of $S$ removed has three components. Hence, if we omit the 3-connectedness of the graph, then Theorem 2 does not hold. (Note that
using other constructions, we can show that Theorem 2 does not hold even if we assume 2-connectedness and “minimum degree at least three”, instead of the 3-connectedness.

We secondly consider the condition for a partite set of 4-vertices. Let $e$ and $e'$ be edges of $K$ which do not lie on the boundary 4-cycle of the same face, and let $K' = K - \{e, e'\}$, where we let $h$ and $h'$ be the hexagonal faces of $K'$ containing $e$ and $e'$ in $K$, respectively. Let $Q$ be the bipartite torus quadrangulation obtained from $K'$ by pasting $T$ to $h$ and $T'$ to $h'$ so that vertices with the same color coincide. Then $Q$ is a balanced 3-connected bipartite torus quadrangulation, but it is not 1-tough. (Removing the six white vertices of $T$ in $Q$, we get at least seven components.) Since $Q$ is not Hamiltonian, the condition for a partite set of 4-vertices cannot be omitted, either.

1-Toughness and embeddability in the torus. Let $G$ be any 4-connected graph on any surface $F^2$ with Euler characteristic $\chi(F^2) \geq 0$. Then $G$ must be 1-tough. (Let $S \subset V(G)$ be any set with $G - S$ disconnected. Let $s = |S|$ and let $w$ denote the number of components of $G - S$. Let $K$ be the graph on $F^2$ obtained from $G$ by contracting each component of $G - S$ into a single vertex and deleting each edge joining two vertices in $S$. Since $K$ is a bipartite graph on $F^2$, we have $|E(K)| \leq 2(s + w) - 2\chi(F^2)$, by Euler’s formula. On the other hand, since $G$ is 4-connected, each component of $G - S$ has at least four edges joined to $S$ in $K$, and hence we have $|E(K)| \geq 4w$. By the two inequalities, we have $w \leq s - \chi(F^2)$, which means that $G$ is 1-tough since $\chi(F^2) \geq 0$.) On the other hand, if we relax the 4-connectivity, those surfaces admit infinitely many 3-connected balanced bipartite graphs which are not 1-tough, as in the previous. However, restricting the vertex degree in one of the partite sets to be four in 3-connected balanced bipartite toroidal graphs, we get the Hamiltonicity of them as in Theorem 3. Therefore, the condition of $Q$ in Theorem 3 is sufficient for $Q$ to be 1-tough, though it seems difficult to prove it directly from the assumption on $Q$. So Corollary 4 is worth mentioning.

**Corollary 4.** Let $Q$ be a 3-connected balanced bipartite graph which is embeddable in the torus. If one of the two partite sets consists only of degree four vertices, then $Q$ is 1-tough.

Here we show that the condition on $Q$ to be embeddable in the torus is necessary in the above corollary (and also in Theorem 3). Let $A$ be three vertices in the same partite set of $K_{3,3}$. Let $B$ be three vertices in the same partite set of a 4-regular bipartite quadrangulation $T$ on the torus which has a unique embedding on the torus. Let $G$ be the graph obtained by joining $A$ and $B$ with a perfect matching. Then $G$ is a 3-connected balanced bipartite graph one of whose partite sets consists only of degree four vertices, but $G$ is not embeddable in the torus since $T$ has a unique embedding on the torus. It is easy to see that $G$ is not 1-tough since $G - A$ has four components.

**Conjecture 1 and Theorem 2.** Using the notion of face subdivisions, we mention that Theorem 2 partially solves Conjecture 1. Let $H$ be a torus quadrangulation with black vertices, which is bipartite or non-bipartite. It is easy to see that its face subdivision $\tilde{H}$ is 4-connected. Hence, if Conjecture 1 is true, then $\tilde{H}$ is Hamiltonian. Moreover, if $\tilde{H}$ has a Hamiltonian cycle $C$, then $C$ passes through black and white vertices alternately in $\tilde{H}$,
since the number of vertices equals that of faces in $H$ by Euler’s formula, and since the white vertices are independent in $\tilde{H}$. Therefore, the radial graph of $H$ is also Hamiltonian and it is nothing but a bipartite torus quadrangulation with each white vertex of degree 4. So the affirmative solution of Conjecture 1 implies Theorem 2. Therefore, Theorem 2 not only solves the Hamiltonicity for a large class of bipartite torus quadrangulations, but the theorem also partially solves Conjecture 1.

4-Connected non-Hamiltonian maps of negative Euler characteristics. Let $Q$ be a quadrangulation on a surface with a negative Euler characteristic. Then, by Euler’s formula, the number of faces of $Q$ is strictly greater than that of vertices, and hence the face subdivision $\tilde{Q}$ of $Q$ is not 1-tough. Therefore, $\tilde{Q}$ is 4-connected but it is not Hamiltonian. Hence, for maps on surfaces with negative Euler characteristics, the 4-connectedness does not necessarily imply the Hamiltonicity. (Yu proved that every 5-connected triangulation on any non-spherical surface with sufficiently large representativity is Hamiltonian [16].)

If we consider the above construction for the torus, then we can get a 4-connected graph on the torus with the toughness exactly 1. (The existence of 4-connected graphs on the torus having the toughness exactly one is a big reason for the difficulty of Conjecture 1.) So Theorem 2 asserts that the face subdivision of any torus quadrangulation has a Hamiltonian cycle, though its toughness is exactly 1.

Hamiltonicity of 4-connected toroidal graphs with toughness exactly one. As mentioned in the previous, there exists 4-connected toroidal graphs with toughness exactly one, which seems to make the problem difficult. However, using Theorem 3, we can prove the following surprising result:

**Corollary 5.** Let $G$ be a 4-connected toroidal graph. If the toughness of $G$ is exactly one, then $G$ is Hamiltonian.

**Proof.** Let $G$ be a 4-connected toroidal graph with the toughness exactly one. Then there exists $S \subset V(G)$ such that $\omega(G - S) = |S|$. Here we take $S$ so that $|S|$ is as small as possible. Let $H$ be the torus map obtained from $G$ by contracting each component of $G - S$ to a single vertex and removing all edges joining two vertices of $S$. Note that $H$ is a balanced bipartite graph of order $|S| + \omega(G - S)$. Since $G$ is 4-connected, we have $|E(H)| \geq 4\omega(G - S)$. Moreover, since $G$ is embeddable in the torus, we have $|E(H)| \leq 2(|S| + \omega(G - S))$, by Euler’s formula. By the fact “$\omega(G - S) = |S|$”, the equalities must hold in the above two inequalities. So any vertex in $H$ corresponding to a component of $G - S$ has degree four, and $H$ is a quadrangular map on the torus.

Suppose that $H$ has distinct two vertices $u_1, u_2$ with $H - \{u_1, u_2\}$ disconnected. Since $H$ is a quadrangular map, $H$ has two faces, say $u_1u_2x$ and $u_1bu_2y$ such that $a$ and $b$ are inner vertices of the plane subgraph bounded by the 4-cycle $u_1xu_2y$. Observe that either $u_1, u_2 \in S$ or $u_1, u_2 \notin S$. In the former, $G - \{u_1, u_2\}$ is disconnected, contrary to the 4-connectedness of $G$. In the latter, we have $a, b, x, y \in S$. If $a \neq b$, then $H$ has two faces $xu_1yc$ and $xu_2yd$ for some $c, d \in V(H)$, since $\deg_H(u_1) = \deg_H(u_2) = 4$. Then $G - \{x, y\}$ is disconnected, contrary to the 4-connectedness of $G$. On the other hand, if $a = b$, then...
putting $S' = S - \{a\}$, we have $\omega(G - S') = \omega(G - S) - 1 = |S| - 1 = |S'|$, contrary to the minimality of $S$. Hence $H$ is 3-connected.

By Theorem 3, $H$ has a Hamiltonian cycle $T$. We shall recover each component of $G - S$ and find a Hamiltonian cycle of $G$.

Let $v \notin S$ be a vertex in $H$ and let $u_1, u_2, u_3, u_4$ be four distinct neighbors of $v$ in $H$ lying in this cyclic order on the torus. Let $C_v$ be a component of $G - S$ corresponding to $v$, and let $R_v$ be the plane graph obtained from the subgraph of $G$ induced by $V(C_v) \cup \{u_1, u_2, u_3, u_4\}$ by adding $u_iu_{i+1}$ for $i = 1, 2, 3, 4$ if $u_iu_{i+1} \notin E(G)$. Thomassen’s result (Main Theorem in [13]) and Thomas and Yu’s result (Lemma (2.4) in [10]) imply that for any distinct $i, j \in \{1, 2, 3, 4\}$, $R - \{u_k, u_l\}$ has a Hamiltonian path $T_v$ connecting $u_i$ and $u_j$, where $\{k, l\} = \{1, 2, 3, 4\} - \{i, j\}$. Hence, when $T$ passes through $v_i, v, v_j$ in this order in $H$, then we can take a corresponding path $T_v$ in $G$. Doing the same procedures for all components of $G - S$, we obtain a Hamiltonian cycle of $G$. □

3 2-Orientations and vertex-faces curves

Let $Q$ be a quadrangulation and let $f$ be a face of $Q$ bounded by a cycle $abcd$. We say that each of $\{a, c\}$ and $\{b, d\}$ is a diagonal pair of $f$ in $Q$. A quadrangular map is a map, possibly with multiple edges, such that each face is bounded by a 4-cycle, and in particular, it is a quadrangulation if the graph is simple. It is known that any quadrangular map is 2-connected and 2-representative.

Proposition 6. Let $Q$ be a bipartite torus quadrangulation with black and white vertices in which every white vertex has degree 4. Then there exists a quadrangular map $H$ such that $R(H) = Q$. In particular, if $Q$ is 3-connected, then $H$ has no contractible 2-cycle.

Proof. First, let $H$ be the graph on the black vertices obtained from $H$ by such a way that for each face of $Q$, we connect two black vertices which form a diagonal pair of the face. Then $R(H) = Q$. It is easy to see that $H$ has no loop. (Otherwise $Q$ has a cut vertex or the representativity at most one, a contradiction.) In particular, every face of $H$ is bounded by a cycle and can be taken as a quadrilateral since each white vertex of $Q$ has degree four.

Suppose that $H$ has a contractible 2-cycle $C = xy$. Then $Q$ has two quadrilateral faces, say $xpyq$ and $xysl$, each of whose diagonal pair is $\{x, y\}$. Since $C$ is contractible on the torus, we can take a contractible simple closed curve $\gamma$ along $C$ intersecting $Q$ only at $x$ and $y$. Hence $Q - \{x, y\}$ is disconnected and hence $Q$ is not 3-connected. □

Let $G$ be a map on a surface $F^2$ and let $L_m = \{l_1, \ldots, l_m\}$ be a set of disjoint simple closed curves on $F^2$. We say that $L_m$ is a vertex-face $m$-family for $Q$ if each vertex and each face of $Q$ is visited exactly once by a member of $L_m$ but every $l_i$ crosses no edge of $Q$ transversely. In particular, when $m = 1$, the unique element of $L_1$ is called a vertex-face curve for $Q$.

The following is an easy observation, since a vertex-face curve passes through vertices and faces alternately.
Proposition 7. Let $G$ be a map on a surface and let $R(G)$ be the radial graph of $Q$. Then $R(G)$ is Hamiltonian if and only if $G$ admits a vertex-face curve.

Hence, by Propositions 6 and 7, the following theorem implies Theorem 2. So, showing Theorem 8 is the main purpose of this paper.

Theorem 8. Every quadrangular map $Q$ on the torus with no contractible 2-cycle admits a vertex-face curve.

In [8], Theorem 8 has already been proved when $Q$ is simple and bipartite. So, in this paper, modifying the argument, we prove Theorem 8 when $Q$ is non-bipartite.

In order to prove Theorem 8, we use an orientation of a graph $G$, that is, an assignment of a direction to each edge of $G$. Let $-\rightarrow G$ denote the graph with the orientation and distinguish it from the undirected graph $G$. For a vertex $v$ of $-\rightarrow G$, the outdegree of $v$ is the number of directed edges outgoing from $v$ and denoted by $\text{od}(v)$. We say that $-\rightarrow G$ is a $k$-orientation or $k$-oriented if each vertex of $-\rightarrow G$ has outdegree exactly $k$. We need the following.

Lemma 9. Every quadrangular map with no contractible 2-cycle on the torus admits a 2-orientation.

The above has been proved only for quadrangulations in [8], and we modify the argument to deal with quadrangular map possibly with multiple edges. Let $G$ be a graph and let $f : V(G) \rightarrow \{0, 1, 2, \ldots\}$ be a function. An orientation of $G$ is called an $f$-orientation if $\text{od}(v) = f(v)$ for any $v \in V(G)$. For $S \subset V(G)$, let $[S]$ denote the subgraph of $G$ induced by $S$.

Proposition 10 ([8]). Let $G$ be a graph and let $f : V(G) \rightarrow \{0, 1, 2, \ldots\}$ be a function. Then $G$ has an $f$-orientation $-\rightarrow G$ if and only if $\sum_{v \in V(G)} f(v) = |E(G)|$, and for any $S \subset V(G)$ with $[S]$ connected, $\sum_{s \in S} f(s) \geq |E([S])|$. ■

We shall prove Lemma 9, using Proposition 10.

Proof of Lemma 9. By Proposition 10, a graph $Q$ with $n$ vertices has a 2-orientation if and only if (1) $|E(Q)| = 2n$, and (2) for any connected subgraph $T$ of $Q$, $|E([S])| \leq 2|V(T)|$. By Euler’s formula, we can verify (1) for $Q$ and so, we prove (2).

We can deal with $T$ as a torus map. Observe that $T$ has no loop, that $T$ has no face bounded by a 2-cycle by the assumption, and that $T$ has no triangular face since $Q$ is a quadrangulation. Hence $T$ has no $k$-face with $k \leq 3$, and so we have $2|E(T)| \geq 4|F(T)|$. By Euler’s formula, $|V(T)| - |E(T)| + |F(T)| \geq 0$. By these two, we have $2|V(T)| \geq |E(T)|$ and we are done. ■

In our proof, we need the following lemma.

Lemma 11. Every quadrangular map $G$ on the torus has an essential even cycle.
Let $G$ and $H$ be two maps on the same non-spherical surface $F^2$. We say that $H$ is a surface-minor of $G$ if $H$ is obtained from $G$ by deletions and contractions of edges on $F^2$. Let $G$ be a $k$-representative map on a non-spherical surface. We say that $G$ is $k$-minimal if $r(G) = k$ and $r(G') < k$ for any proper surface-minor $G'$ of $G$. It is known that for any non-spherical surface $F^2$ and any fixed integer $k \geq 1$, there exist only finitely many $k$-minimal maps on $F^2$, up to homeomorphism. In particular, the complete list for $2$-minimal torus maps has been determined in [7].

**Theorem 12 (Nakamoto [7]).** There exist exactly seven $2$-minimal map on the torus, which are $T_1, \ldots, T_7$ listed in Figure 2, in which each rectangle expresses the torus by identifying the top and the bottom, and the right and the left sides, respectively.

![Figure 2: The 2-minimal maps on the torus](image.png)

**Proof of Lemma 11.** Since $G$ is $2$-representative, $G$ can be transformed by deletions and contractions of edges into one of $T_1, \ldots, T_7$ in Figure 2, by Theorem 12. Let $B$ be a torus map with a single vertex $v$ and three pairwise non-homotopic loops $e_1, e_2, e_3$. We can verify that each $T_i$ has $B$ as a surface-minor. Hence, each $T_i$ has an essential cycle.
contracted to $e_j$, for $j = 1, 2, 3$, since contractions of edges preserve homotopy types of cycles on the surface. Moreover, since some $T_i$ is a surface-minor of $G$, $G$ also has an essential cycle $C_j$ contracted to $e_j$, for $j = 1, 2, 3$.

It is easy to see that in a quadrangular map, two homotopic closed walks have the same parity of length. Suppose that $C_1$ and $C_2$ have odd length. (Otherwise we are done.) Now cutting the torus where $B$ embeds along $e_1$ and $e_2$, we get a rectangle with a diagonal $e_3$. Hence $e_3$ is homotopic to the concatenation of $e_1$ and $e_2$. Since $C_1$ and $C_2$ are non-homotopic on the torus, then they have a common vertex $v$ in $G$. Let $W$ be a closed walk starting at $v$, proceeding along $C_1$ and return to $v$, and proceeding along $C_2$ and stopping at $v$. Then $W$ is an essential closed walk of even length, since $C_1$ and $C_2$ are non-homotopic essential odd cycles. Clearly, $C_3$ is an essential cycle homotopic to $W$, and hence its length must be even.

4 Converting $Q$ into a bipartite graph

In this paper, we have to prove Theorem 8. In order to do so, we often cut the torus along a simple essential closed curve $l$ to convert a non-bipartite torus quadrangulation into a bipartite torus quadrangulation. Let $G$ be a map on a surface $F^2$ and suppose that each edge of $G$ lies on $l$ or intersects $l$ at its endpoints. Cutting $G$ along $l$ is to cut $F^2$ along $l$ so that in the map on the resulting surface, each vertex and each edge of $G$ lying on $l$ appears twice on the boundary. When $F^2$ is the torus, the map the after cutting is one on the annulus.

Let $Q$ be a non-bipartite torus quadrangular map. Then, by Lemma 11, $Q$ has an essential even cycle $C = v_1v_2\ldots v_p$. Let $Q_C$ be the annulus quadrangular map obtained from $Q$ by cutting along $C$. Let $Y = y_1y_2\ldots y_p$ and $Z = z_1z_2\ldots z_p$ be the two boundary cycles of $Q_C$, where $y_i$ and $z_i$ correspond to $v_i$ in $Q$ for each $i$. Since $Q$ is a quadrangular map and $C$ is an essential even cycle, $Q_C$ is a bipartite graph. Let $Q'_C$ be the bipartite torus quadrangular map naturally obtained from $Q_C$ by adding the edges $y_iz_i$ for $i = 1, \ldots, p$.

In order to prove Theorem 8, the following is the most important argument in this paper, which will be proved in the last section.

**Theorem 13.** Let $Q$ be a non-bipartite quadrangular map on the torus with no contractible 2-cycle. Then an essential even cycle $C = v_1\ldots v_p$ can be chosen in $Q$ so that the bipartite quadrangular map $Q'_C$ has a 2-orientation which has oriented edges

- $y_{i+1}y_i$, $z_iz_{i+1}$ for any integer $i$, (the index is taken modulo $p$)
- $z_iz_i$ for any odd integer $i$, and
- $y_iz_i$ for any even integer $i$.

Before proceeding to the next section, we show that Theorem 13 proves Theorem 8, as follows.

**Proposition 14.** If Theorem 13 holds, then does Theorem 8.
Proof. Let $Q$ be a quadrangular map on the torus with no contractible 2-cycle. If $Q$ is bipartite, then let $G = Q$. Otherwise we take an essential even cycle $C$ and let $G = Q_C$. By Lemma 9 (when $G = Q$) or by Theorem 13 (when $G = Q_C$), $G$ has a 2-orientation and let $\overrightarrow{G}$ be the oriented graph. When $Q$ is non-bipartite, we take a 2-orientation of $G$ satisfying the conclusion of Theorem 13.

Since $G$ is the bipartite graph, let $B$ and $W$ be the partite sets of $G$. Since the torus is orientable, we can give a clockwise orientation at each point on the torus simultaneously. Suppose that $\{e_1, \ldots, e_m\}$ be the edges of $\overrightarrow{G}$ incident to a vertex $v$, where $\deg_G(v) = m$ and $\{e_1, \ldots, e_m\}$ appear around $v$ in this clockwise order. Without loss of generality, we may assume that $e_1$ and $e_k$ are two outgoing edges on $v$. Now we can put a segment $l_v$ through a vertex $v$ which locally separates all edges incident to $v$ into two sets as shown in Figure 3:

- if $v \in B$, then $e_1 \ldots e_{k-1}$ are located in one of the two sides separated by $l_v$, and all others are in the other side of $l_v$;
- if $v \in W$, then $e_2 \ldots e_k$ are located in one of the two sides separated by $l_v$, and all others are in the other side of $l_v$.

Next we consider whether we can glue $l_v$’s for all $v \in V(\overrightarrow{G})$ to get a vertex-face $m$-family for some $m \geq 1$. Let us consider a quadrilateral face $f$ of $\overrightarrow{G}$ bounded by a 4-cycle $bwb'w'$, where $b, b' \in B$ and $w, w' \in W$. Observe that the 4-cycle $bwb'w'$ has seven distinct orientations shown in Figure 4, up to symmetry. In each case, by the definition of $l_v$’s, we can find that exactly two segments intersect at each face, and glue them at a center of the face. Hence $\bigcup_{v \in V(\overrightarrow{G})} l_v$ form a set of several simple closed curves visiting each vertex and each face of $\overrightarrow{G}$ exactly once, but crossing no edge of $\overrightarrow{G}$ transversely. Therefore, $\bigcup_{v \in V(\overrightarrow{G})} l_v$ can be regarded as a vertex-face $m$-family $L_m = \{l_1, \ldots, l_m\}$ for some $m \geq 1$.

In [8], it is proved that each $l_i$ is essential in the torus. Then we can take the indices of $l_1, \ldots, l_m$ so that $l_i$ and $l_{i+1}$ bounds an annulus $A_i$ not containing any other $l_j$. Moreover, when $m \geq 2$, it has been proved that for any $i$, changing the orientation of some edges in $A_i$, we can take a simple closed curve $l'_i$ such that $\{l_1, \ldots, l_{i-1}, l'_i, l_{i+2}, \ldots, l_m\}$ is a vertex-face $(m-1)$-family in the new 2-orientation of $G$. Therefore, if $G = Q$, then by iteration of changing the orientation, we can find a vertex-face curve for $Q$. 

![Figure 3: Segments for vertices](image1)

![Figure 4: Segments for faces](image2)
Now consider the case where $G = Q'_C$. Let $A$ be the annulus bounded by $Y$ and $Z$ and containing the edges $y_i z_i$'s. Then, by the rules for the arrangement of segments and the way to glue them, $A$ contains a single simple closed curve, say $l_1$, in a vertex-face $m$-family \{$l_1, \ldots, l_m$\} for some $m \geq 1$, passing through all $z_i$ with $i$ odd, all $y_i$ with $i$ even and no others. See Figure 5. Since $Q'_C$ has faces not contained in $A$, we have $m \geq 2$. If $m \geq 3$, then we can merge $l_2$ and $l_3$ to get a vertex-face $(m - 1)$-family containing $l_1$. If $m = 2$, then contracting all edges $y_i z_i$ for $i = 1, \ldots, m$ to eliminate $l_1$, we get a vertex-face curve for $Q$ as shown in Figure 6, since $l_2$ passes through all vertices and all faces of $Q$ exactly once.

5 Proof of the theorems

We first prove Theorem 13.

Proof of Theorem 13. If $C$ is a cycle and $P$ is a path which meets $C$ exactly in its end-vertices, then we say that $P$ is a $C$-path. For an essential cycle $C$, a $C$-path $P$ is called an essential $C$-path if no cycle in $C \cup P$ bounds a 2-cell on $F^2$. Especially in case of $|E(P)| = 1$, we say that $P$ is an essential $C$-edge. For two vertices $x$ and $y$ in a graph
Let $C$ be the set of shortest essential even cycles of $Q$. By Lemma 11, $C \neq \emptyset$. Let $C = v_1 v_2 \ldots v_p$ be a cycle in $C$. In this section, we always consider the subscript of $v$ as modulo $p$. Since $Q$ is a quadrangular map, it follows that

\[
\text{if } v_i v_j \in E(Q) \setminus E(C), \text{ then } v_i v_j \text{ is an essential } C \text{-edge.}
\]  

(1)

Moreover, since $Q$ is a non-bipartite quadrangular map and $C$ is an essential even cycle, the cycle $v_i v_{i+1} \ldots v_j v_i$ is an odd cycle. (For otherwise, i.e., if it is even, then cutting along $C \cup \{v_i v_j\}$, we get a disk with even length boundary from the torus. Then a bipartition of $C \cup \{v_i v_j\}$ can be extended to that of $G$, contrary to the non-bipartiteness of $Q$.) Thus the parities of $i$ and $j$ are the same.

Here we fix an orientation of $C$ and let $E_R(C, v_i)$ and $E_L(C, v_i)$ denote the set of edges in $E(Q) \setminus E(C)$ which are adjacent to $v_i$ on the right-hand side and left-hand side of $C$, respectively. We define six sets of pairs of indices as follows:

\[
\Lambda_B(C) = \{(i, j) \mid i, j: \text{odd}, v_i v_j \in E_L(C, v_i), v_j v_i \in E_L(C, v_j) \text{ for some } v \in V(C)\} \cup \\
\{(i, j) \mid i, j: \text{even}, v_i v_j \in E_R(C, v_i), v_j v_i \in E_R(C, v_j) \text{ for some } v \in V(C)\},
\]

\[
\Lambda_W(C) = \{(i, j) \mid i, j: \text{odd}, v_i v_j \in E_R(C, v_i), v_j v_i \in E_R(C, v_j) \text{ for some } v \in V(C)\} \cup \\
\{(i, j) \mid i, j: \text{even}, v_i v_j \in E_L(C, v_i), v_j v_i \in E_L(C, v_j) \text{ for some } v \in V(C)\},
\]

\[
\lambda_B(C) = \{(i, j) \mid i, j: \text{odd}, v_i v_j \in E_L(C, v_i), v_j v_i \in E_L(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\} \cup \\
\{(i, j) \mid i, j: \text{even}, v_i v_j \in E_R(C, v_i), v_j v_i \in E_R(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\},
\]

\[
\lambda_W(C) = \{(i, j) \mid i, j: \text{odd}, v_i v_j \in E_R(C, v_i), v_j v_i \in E_R(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\} \cup \\
\{(i, j) \mid i, j: \text{even}, v_i v_j \in E_L(C, v_i), v_j v_i \in E_L(C, v_j) \text{ for some } v \in V(Q) \setminus V(C)\},
\]

\[
X_B(C) = \{(i, j) \mid i: \text{even}, j: \text{odd}, v_i v_j \in E_L(C, v_i), v_{i+2} v_j \in E_L(C, v_{i+2}), \\
v_j v_i \in E_R(C, v_j) \text{ and } v_{j+2} v_i \in E_R(C, v_{j+2}) \text{ for some } v \in V(Q) \setminus V(C)\};
\]

\[
X_W(C) = \{(i, j) \mid i: \text{even}, j: \text{odd}, v_i v_j \in E_L(C, v_i), v_{i+2} v_j \in E_L(C, v_{i+2}), \\
v_j v_i \in E_R(C, v_j) \text{ and } v_{j+2} v_i \in E_R(C, v_{j+2}) \text{ for some } v \in V(Q) \setminus V(C)\}.
\]

By the minimality of $C$, $d_C(v_i, v_j) = 2$ holds for every $(i, j) \in \Lambda_B(C) \cup \Lambda_W(C)$, that is, $j = i + 2$ or $i = j + 2$. For $(i, i + 2) \in \lambda_W(C)$, we take $v_{i+1}' \in N(v_i) \cap N(v_{i+2})$ such that $v_i v_{i+2} v_{i+2}' v_{i+1}' v_i$ bounds a 2-cell so that $v_i v_{i+1}'$ is the rightmost edge when $i$ is odd, and $v_i v_{i+1}'$ is the leftmost edge when $i$ is even. Let

\[
C' = v_1 v_2 \ldots v_i v_{i+1}' v_{i+2} v_{i+3} \ldots v_1.
\]  

(2)

When $i$ is odd (resp. even), there is no common neighbor of $v_i$ and $v_{i+2}$ in the right-hand (resp. left-hand) side of $C'$ by the choice of $v_{i+1}'$, and there is no common neighbor of $v_{i+1}'$ and $v_{i+3}$ in the left-hand (resp. right-hand) side of $C'$ since every edge in $E_L(C', v_{i+1}')$ is contained in the 2-cell bounded by $v_i v_{i+1}' v_{i+2} v_{i+1}$. Therefore, the following hold:

\[
\lambda_W(C') \subseteq \lambda_W(C) \setminus \{(i, i + 2)\},
\]  

(3)

\[
X_W(C') \subseteq X_W(C) \text{ and}
\]  

(4)

\[
X_W(C) = \emptyset \Rightarrow \Lambda_W(C') \subseteq \Lambda_W(C).
\]  

(5)
Here we may assume that $C \in \mathcal{C}$ was chosen so that:

(i) $|\lambda_W(C)|$ is as small as possible and

(ii) $|\Lambda_W(C)|$ is as small as possible, subject to (i).

Now we prove $\Lambda_W(C) = \lambda_W(C) = \emptyset$. If $(i, i + 2) \in \lambda_W(C)$, we can take the new cycle $C'$ as in (2). Then it follows from (3) that $|\lambda_W(C')| < |\lambda_W(C)|$, which is a contradiction. Hence $\lambda_W(C) = \emptyset$ holds.

Next we shall prove $\Lambda_W(C) = \emptyset$. We need the following claim.

Claim 1. Let $v_s, v'_s$ be an essential $C$-edge and $P$ be an essential $C$-path joining $v_t$ and $v'_t$, where $s \neq t$, $s' \neq t'$ and $t \neq t'$. Then

(I) If $v_s, v_t, v'_t$ appear in $C$ in this order or in the order $v_s, v'_t, v_t, v_t$ (possibly, $s = t'$ and/or $s' = t$) then $d_C(v_s, v_t) + d_C(v'_t, v_t) \leq |E(P)| + 1$.

(II) If $v_s, v_t, v'_t, v'_t$ appear in $C$ in an order other than (I), then $d_C(v_s, v'_t) \leq |E(P)|$.

Proof. Assume, to the contrary, that $d_C(v_t, v) + d_C(v'_t, v'_t) \geq |E(P)| + 2$ in the case (I) or $d_C(v_s, v'_s) \geq |E(P)| + 1$ in the case (II). Without loss of generality, we may assume that $s \leq s', t, t'$. Here we define an essential even cycle $C'$ as follows:

(I-1) if $t \leq s' < t'$, then let $C' = v_1v_2 \ldots v_sv_{s'-1}v_tPv_tv_{t+1} \ldots v_1$;

(I-2) if $t' < s' \leq t$, then let $C' = v_s^tv_{s+1} \ldots v_tPv_{t-1} \ldots v_s^tv_s$;

(II-1) if $s' < t < t'$, then let $C' = v_1v_2 \ldots v_sv_{s'-1}v_tPv_tv_{t+1} \ldots v_1$;

(II-2) if $s' < t' < t$, then let $C' = v_1v_2 \ldots v_sv_{s'+1}v_tPv_tv_{t+1} \ldots v_1$;

(II-3) if $t < t' < s'$, then let $C' = v_s^sv_{s+1} \ldots v_tPv_{t+1} \ldots v_s^tv_s$;

(II-4) if $t' < t < s'$, then let $C' = v_s^sv_{s+1} \ldots v_tPv_{t+1} \ldots v_s^tv_s$.

(See Figure 7.) Then

$$|C'| \leq \begin{cases} |C| - d_C(v_s, v_t) - d_C(v'_t, v_t) + |E(P)| + 1 & \text{in the case (I-1) and (I-2)}, \\ |C| - d_C(v_s, v'_s) - d_C(v_t, v'_t) + |E(P)| + 1 & \text{in the case (II-1) - (II-4)}. \end{cases}$$

Since $d_C(v_t, v) + d_C(v'_t, v'_t) \geq |E(P)| + 2$ in the case (I-1) and (I-2), and since $d_C(v_s, v'_s) \geq |E(P)| + 1$ and $d_C(v_t, v'_t) \geq 1$ in the case (II-1) - (II-4), $C'$ is shorter than $C$. Thus we obtain a contradiction in each case. $\square$

Assume $\Lambda_W(C) \neq \emptyset$. Then, without loss of generality, we may assume that there exist three vertices $v_k, v_l$ and $v_{k'}$ in $V(C)$ such that $v_kv_k' \in E_R(C, v_k)$ and $v_lv_l \in E_R(C, v_l)$ for some odd integers $k$ and $l$ with $k < l$. Also we may assume that the cycle $v_kv_{k+1} \ldots v_{l-1}v_l$ bounds a 2-cell on the torus. Since $Q$ has no loop, $v_k, v_l$ and $v_{k'}$ are distinct. If $k <$
$k' < l$, then $v_1v_2\ldots v_kv_{k'}v_{k+1}\ldots v_1$ is an essential even cycle which is shorter than $C$, a contradiction. Hence $k' < k$ or $l < k'$ holds. Without loss of generality, we may assume that $l < k'$. Then we have $k' \geq k + 4$ and $k' \geq l + 2$.

Using Claim 1, we consider possible positions of essential $C$-edges and essential $C$-paths, in order to prove $\Lambda_B(C) = \emptyset$ and $X_B(C) = \emptyset$ in the following two paragraphs respectively.

Suppose that there exists an essential $C$-edge $v_mv_{m'}$ with $v_mv_{m'} \in E_R(C, v_m)$ and $m' \neq k'$. Recall that the parities of $m$ and $m'$ are the same. Now consider Claim 1 for $s = k$ (or $s = l$), $s' = k'$, $t = m$ and $t' = m'$. Since $d_C(v_k, v_{k'}) \geq 2$ and $d_C(v_l, v_{k'}) \geq 2$, the case (II) does not occur for both cases, and hence $k = m$ or $d_C(v_k, v_m) + d_C(v_{k'}, v_{m'}) \leq 2$, and $l = m$ or $d_C(v_l, v_m) + d_C(v_{k'}, v_{m'}) \leq 2$. Since $k' \neq m'$, in either case, we have $d_C(v_k, v_{m'}) \leq 1$ and $d_C(v_l, v_{m'}) \leq 1$. If $m$ is odd, then $d_C(v_k, v_m) = d_C(v_l, v_m) = 0$, contradicting $v_k \neq v_l$.

If $m$ is even, then $d_C(v_k, v_m) = d_C(v_l, v_m) = 1$. Since the cycle $v_kv_{k+1}\ldots v_kv_{k'}v_k$ bounds a 2-cell on the torus, we have $m \leq k$ or $m \geq l$, and hence $m = k - 1 = l + 1$. However, this implies that $k = 1$ and $l = p - 1$, and hence $k < k' < l$, a contradiction. These implies that there exists no essential $C$-edge $v_mv_{m'}$ with $v_mv_{m'} \in E_R(C, v_m)$ and $m' \neq k'$, and hence $\Lambda_B(C) = \emptyset$.

Next suppose that there exists an essential $C$-path $v_mv_{m'}$ of length 2 such that $v_mv \in E_R(C, v_m)$, $m$ is even, and $m' \neq k'$. Note that the parities of $m$ and $m'$ are different, so $d_C(v_k, v_{m'}) \geq 2$. Notice also that $|C| \geq 6$ because $k, l$ and $k'$ are pairwise distinct odd integers. Since the cycle $v_kv_{k+1}\ldots v_kv_{k'}$ bounds a 2-cell on the torus, we have $m < k$ or $m > l$. Now consider Claim 1 for $s = k$ (or $s = l$), $s' = k'$, $t = m$ and $t' = m'$. If the case (I) occurs for the case $s = k$, then $d_C(v_k; v_m) + d_C(v_{k'}, v_{m'}) \leq 3$. Since $k$ is odd and $m$ is even, we have $m = k - 1$ and $m' = k' - 2$ or $k' + 2$; Similarly, if the case (I) occurs for the case $s = l$, then $m = l + 1$ and $m' = k' - 2$ or $k' + 2$. Assume that the case (II) occurs for both cases $s = k$ and $s = l$. Then $d_C(v_k, v_m) \leq 2$ and $d_C(v_{k'}, v_{m'}) \leq 2$, that is, $k' = k - 2 = l + 2$.

Since $m < k$ or $m > l$, we obtain $m = k' \pm 1$. If $m' \neq k' - 2, k' + 2$, then $v_k, v_{m'}, v_{k'}, v_m$ (when $m = k' + 1$) or $v_l, v_m, v_{k'}, v_{m'}$ (when $m = k' - 1$) appear in $C$ in this order, and the case (I) holds, a contradiction. Thus, we have $m' = k' - 2$ or $k' + 2$. In either case,
such that $S$ and $Y$ of black and white and $m_l = k - 2$ or $k' + 2$. This implies that if $(m, m') \in X_B(C)$, then $m = l + 1$, $m + 2 = k - 1$ and $m' = k' - 2$ or $m' = k'$. In this case, $k' = k - 2 = l + 2$. Let $v$ be the vertex such that $vv_n, vv_{n+2}, vv_{n'}, vv_{n'+2} \in E(Q)$. In either case, $v_l v_n v_k v_{k+2}$ is an essential even cycle of length four, contradicting the choice of $C$. Thus, $X_B(C) = \emptyset$.

By changing the parities of the indices of $C$, we can find the cycle $C_1$ with $\Lambda_w(C_1) = X_B(C_1) = \emptyset$. If $\Lambda_w(C_1) = \emptyset$, then $C_1$ is a desired cycle. If $\Lambda_w(C_1) \neq \emptyset$, then it follows from (3) that we can decrease the value $|\Lambda_w(C_1)|$ by taking a new cycle as in (2). Iterating this operation, we obtain an essential even cycle $C'$ such that $\Lambda_w(C') = \emptyset$. Moreover, since $X_B(C_1) = \emptyset$, it follows from (4) and (5) that $\Lambda_w(C_1') \subseteq \Lambda_w(C_1) = \emptyset$. Consequently we have $\Lambda_w(C) = \emptyset$.

Now we remember the annulus quadrangulation $Q_C$ appeared in Section 4, which is obtained from $Q$ by cutting along $C$. Let $Y = yi_1 y_i \ldots y_p$ and $Z = z_1 z_2 \ldots z_p$ be the cycles corresponding to two boundary components of $Q_C$, where $yi$ and $zi$ correspond to the same vertex $v_i$ in $Q$, for each $i$. We may assume that $E_{Q_C}(yi) = E(Y) = E_L(C, y_i)$ for any $yi \in Y$, where $E_{Q_C}(yi)$ is the set of edges incident with $yi$ in $Q_C$. Since $Q$ is a quadrangulation and $C$ is an essential even cycle, $Q_C$ can be regarded as a map on the annulus with each face bounded by an even cycle. Hence $Q_C$ is a bipartite graph. Let

$$Y_W = \bigcup_{i: \text{even}} y_i, \quad Y_B = \bigcup_{i: \text{odd}} y_i, \quad Z_W = \bigcup_{i: \text{odd}} z_i, \quad Z_B = \bigcup_{i: \text{even}} z_i$$

and let

$$\varphi(v) = \begin{cases} 
0 & \text{for } v \in Y_W \cup Z_W \\
1 & \text{for } v \in Y_B \cup Z_B \\
2 & \text{for } v \notin Y \cup Z 
\end{cases}$$

Let $G = Q_C - (E(Y) \cup E(Z))$. Let $W$ and $B$ be a bipartition of $V(G)$, referred as white and black vertices, where we suppose $Y_W, Z_W \subset W$ and $Y_B, Z_B \subset B$. By (1), each of $Y$ and $Z$ is an independent set in $G$.

**Claim 2.** There exists an orientation of $G$ such that $od(v) = \varphi(v)$ for any $v \in V(G)$.

**Proof.** By Euler’s formula, we have $|E(Q)| = 2|V(Q)|$. Thus $|E(G)| = |E(Q_C)| - 2p = |E(Q)| - p = 2|V(Q)| - p = 2|V(Q_C)| - 3p = 2|V(G)| - 3p = \sum_{v \in V(G)} \varphi(v)$. By Proposition 10, it suffices to prove $|E([S])| \leq \sum_{v \in S} \varphi(v)$ for every $S \subset V(G)$ with $|S|$ connected.

**Case 1.** $|S|$ contains an essential cycle on the annulus.

Observe that $|S|$ has two boundary walks $L_1 = a_1 a_2 \ldots a_l a_1$ and $L_2 = b_1 b_2 \ldots b_l b_1$ such that $S \cap Y \subset L_1$ and $S \cap Z \subset L_2$. Let $Y_W' = Y_B \cap L_1$, $Y_W' = Y_W \cap L_1$, $Z_B' = Z_B \cap L_2$ and $Z_W' = Z_W \cap L_2$. Note that $l_1 \geq \{a_i, a_{i+1} \mid a_i \notin Y_B\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} \mid a_i \in Y_W\}$. Since $Y$ is independent, $a_{i+1} \notin Y_B$ holds for every $a_i \in Y_B$. Moreover, for every $a_i \in Y_W$, $\{a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}\} \cap Y_B' = \emptyset$ holds since $G$ is bipartite and $Y$ is independent,
and \{a_{i+1}, a_{i+2}, a_{i+3}\} \cap Y'_W = \emptyset holds since \Lambda_W(C) = \lambda_W(C) = \emptyset. Therefore we have 
l_1 \geq 2|Y'_B| + 4|Y'_W|. Similarly, \(l_2 \geq 2|Z'_B| + 4|Z'_W|\) holds. Thus by Euler’s formula,
\[
|E([S])| \leq 2|S| - \frac{l_1 + l_2}{2} \\
\leq 2|S| - (|Y'_B| + |Z'_B| + 2(|Y'_W| + |Z'_W|)) \\
= \sum_{v \in S} \varphi(v).
\]

**Case 2.** \([S]\) contains no essential cycle on the annulus.

Let \(L = a_1 a_2 \ldots a_t a_1\) be the boundary closed walk of \([S]\), where we note \(S \cap (Y \cup Z) \subset L\). Let \(Y'_B = Y_B \cap L, Y'_W = Y_W \cap L, Z'_B = Z_B \cap L\) and \(Z'_W = Z_W \cap L\). Here we choose two subwalks \(L_Y = a_1 \ldots a_t, L_Z = a_{s+1} \ldots a_t\) of \(L\) so that \(L_Y\) and \(L_Z\) are as short as possible subject to \(1 \leq r < s \leq t, L \cap Y \subset L_Y\) and \(L \cap Z \subset L_Z\).

Since \(Y\) is independent, \(a_{i+1} \notin Y'_B\) holds for every \(a_i \in Y'_B\). Moreover, for every \(a_i \in Y'_W, \{a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}\} \cap Y'_B = \emptyset\) holds since \(G\) is bipartite and \(Y\) is independent, and \(\{a_{i+1}, a_{i+2}, a_{i+3}\} \cap Y'_W = \emptyset\) holds since \(\Lambda_W(C) = \lambda_W(C) = \emptyset\). Now the vertices in \(\{a_i, a_{i+1} | a_i \in Y'_B\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} | a_i \in Y'_W\}\) are contained in

- the segment \(a_1 a_2 \ldots a_r a_{r+1} a_{r+2}\) of \(L\) when \(a_1, a_r \in Y'_W\);
- the segment \(a_1 a_2 \ldots a_r a_{r+1}\) of \(L\) when \(a_1 \in Y'_W\) and \(a_r \notin Y'_W\);
- the segment \(a_1 a_2 \ldots a_r a_{r+1} a_{r+2}\) of \(L\) when \(a_r \in Y'_W\) and \(a_1 \notin Y'_W\);
- the segment \(a_1 a_2 \ldots a_r a_{r+1}\) of \(L\) when \(a_1, a_r \notin Y'_W\).

Hence we have \(r + 1 + |\{a_1, a_r\} \cap Y'_W| \geq |\{a_1, a_{i+1} | a_i \in Y'_B\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2} | a_i \in Y'_W\}| = 2|Y'_B| + 4|Y'_W|,\) and thus \(r \geq 2|Y'_B| + 4|Y'_W| - 1 - |\{a_1, a_r\} \cap Y'_W|\) holds. Similarly we obtain \(t - s + 1 \geq 2|Z'_B| + 4|Z'_W| - 1 - |\{a_s, a_t\} \cap Z'_W|\). Consequently,
\[
l = r + (t - s + 1) + (l - t) + (s - 1 - r) \\
\geq 2|Y'_B| + 4|Y'_W| - 1 - |\{a_1, a_r\} \cap Y'_W| \\
\quad + 2|Z'_B| + 4|Z'_W| - 1 - |\{a_s, a_t\} \cap Z'_W| + (l - t) + (s - 1 - r).
\]

Here \(l - t \geq 1\) when \(\{|a_1\} \cap Y'_W| + |\{a_t\} \cap Z'_W| = 2,\) and \(s - 1 - r \geq 1\) when \(|\{a_r\} \cap Y'_W| + |\{a_s\} \cap Z'_W| = 2,\) Hence \(-|\{a_1, a_r\} \cap Y'_W| - |\{a_s, a_t\} \cap Z'_W| + (l - t) + (s - 1 - r) \geq -2,\) which implies \(l \geq 4(|Y'_B| + |Z'_W|) + 2(|Y'_B| + |Z'_W|) - 4.\) Thus by Euler’s formula,
\[
|E([S])| \leq 2|S| - l/2 - 2 \\
\leq 2|S| - 2(|Y'_W| + |Z'_W|) + |Y'_B| + |Z'_B| \\
= \sum_{v \in S} \varphi(v).
\]

This completes the proof of Claim 2. □
Consider the orientation of $Q_C$ which is obtained from the orientation of $G$ constructed in Claim 2 and oriented edges $y_{i+1}y_i, z_iz_{i+1}$ for any integer $i$, $zy_i$ for any odd integer $i$, and $yz_i$ for any even integer $i$ for this direction. Then this is an orientation of $Q_C$ required in the lemma, and hence we are done.

Finally, we shall prove Theorem 2.

**Proof of Theorem 2.** Let $Q$ be a 3-connected bipartite torus quadrangulation one of whose partite set consists of degree four vertices. By Proposition 6, there exists a quadrangular map $H$ with no contractible 2-cycle such that the radial graph of $H$ is isomorphic to $Q$. Now observe that Theorem 8 has been proved by Theorem 13 and Proposition 14. So $H$ has a vertex-face curve, and hence, by Proposition 7, $Q$ is Hamiltonian.

**References**


